Constrained Multivariate Optimality and Duality

Antonio Frangioni

Department of Computer Science
University of Pisa
https://www.di.unipi.it/~frangio
mailto:frangio@di.unipi.it

Computational Mathematics for Learning and Data Analysis
Master in Computer Science – University of Pisa

A.Y. 2024/25

Outline

- Finally back to the full (P) $f_* = \min\{\, f(x) \,:\, x \in X \,\}$, $X \subset \mathbb{R}^n$
- "Abstract" constraint $x \in X$, implementation discussed later
- $x_* \in X$ s.t. $f(x_*) \leq f(x) \; \forall\, x \in X$: optimal solution $\equiv$ global optimum
- Constraints can be hidden in the objective: $\imath_X : \mathbb{R}^n \to \bar{\mathbb{R}}$ indicator function of X

$$\imath_X(x) = \begin{cases} 0 & \text{if } x \in X \\ \infty & \text{if } x \notin X \end{cases} \qquad (\text{convex} \iff X \text{ is, but extended-valued})$$

$$\implies (P) \equiv \min\{\, f_X(x) = f(x) + \imath_X(x) \,\} \quad \text{(essential objective)}$$

- A very bad idea: $\imath_X \notin C^0 \implies$ ferociously $\notin C^1$
- Conversely, objective "complex" $\to$ "simple" by "hiding it in the constraints"

$$(P) \equiv \min\{\, v \,:\, v \geq f(x) \,,\; x \in X \,\} \quad \text{(a trick we have} \approx \text{seen already)}$$

- Sometimes useful, but "nonlinear objectives easier than nonlinear constraints"

- Note that $X = \emptyset \Longrightarrow v(P) = +\infty\,(= \inf\,\emptyset)$: solving (P) three $\neq$ things
  i) ... ii) ... iii) constructively proving $X = \emptyset$ (how??)

- (Almost) never happens in ML, so we will forget about it (but the issue $\exists$):
  the model is our choice, we choose it simple & nice & nonempty if we can

- Note that $X = \emptyset \Longrightarrow v(P) = +\infty\,(= \inf\ \emptyset)$: solving (P) three $\neq$ things
  i) ... ii) ... iii) constructively proving $X = \emptyset$ (how??)

- (Almost) never happens in ML, so we will forget about it (but the issue $\exists$): the model is our choice, we choose it simple & nice & nonempty if we can

- Global optimum obviously hard, hence x_* local optimum $\equiv$ solves
  $\min\{\, f(\,x\,) \;:\; x \in \mathcal{B}(\,x_*\,,\,\varepsilon\,) \cap X\,\}$ for some $\varepsilon > 0$

- Note that $X = \emptyset \Longrightarrow v(P) = +\infty\,(= \inf\,\emptyset)$: solving (P) three $\neq$ things
  i) … ii) … iii) constructively proving $X = \emptyset$ (how??)

- (Almost) never happens in ML, so we will forget about it (but the issue $\exists$):
  the model is our choice, we choose it simple & nice & nonempty if we can

- Global optimum obviously hard, hence x_* local optimum $\equiv$ solves
  $\min\{\, f(x) \;:\; x \in \mathcal{B}(x_*, \varepsilon) \cap X \,\}$ for some $\varepsilon > 0$

- Important concept: $x \in \mathrm{int}(X)$ (interior) $\equiv$ $\exists\, \mathcal{B}(x, \varepsilon) \subseteq X$ $(\varepsilon > 0)$

- $x_* \in \mathrm{int}(X) \Longrightarrow$ local optimum $\equiv$ local minimum $\Longrightarrow \nabla f(x_*) = 0$

- Constrained (local) optimality conditions $\neq$ "$\nabla f(x) = 0$"
  only if $x \notin \mathrm{int}(X) \iff x \in \partial X$ (boundary)

- Note that $X = \emptyset \Longrightarrow v(P) = +\infty\,(= \inf\,\emptyset)$: solving (P) three $\neq$ things
  i) ... ii) ... iii) constructively proving $X = \emptyset$ (how??)

- (Almost) never happens in ML, so we will forget about it (but the issue $\exists$): the model is our choice, we choose it simple & nice & nonempty if we can

- Global optimum obviously hard, hence x_* local optimum $\equiv$ solves
  $\min\{\, f(x) \;:\; x \in \mathcal{B}(x_*, \varepsilon) \cap X \,\}$ for some $\varepsilon > 0$

- Important concept: $x \in \mathrm{int}(X)$ (interior) $\equiv \exists\,\mathcal{B}(x, \varepsilon) \subseteq X$ $(\varepsilon > 0)$

- $x_* \in \mathrm{int}(X) \Longrightarrow$ local optimum $\equiv$ local minimum $\Longrightarrow \nabla f(x_*) = 0$

- Constrained (local) optimality conditions $\neq$ "$\nabla f(x) = 0$"
  only if $x \notin \mathrm{int}(X) \Longleftrightarrow x \in \partial X$ (boundary)

- Concept intimately tied with X closed / open (glanced about previously)

- Given $S \subseteq \mathbb{R}^n$, interior/boundary points of S:
  - $x \in \text{int}(S) \equiv$ interior of S $\equiv \exists\, r > 0$ s.t. $\mathcal{B}(x, r) \subseteq S$
  - $x \in \partial S \equiv$ boundary of S $\equiv \forall\, r > 0\, \exists\, w, z \in \mathcal{B}(x, r)$ s.t. $w \in S \wedge z \notin S$

  note: $x \in \text{int}(S) \implies x \in S$, but $x \in \partial S \not\Longrightarrow x \in S$

- S open if $S = \text{int}(S)$: "I have no points on the boundary"
- $\text{cl}(S) \equiv$ closure of S $\equiv \text{int}(S) \cup \partial S$: "me and my boundary"
- $S \subseteq \mathbb{R}^n$ closed if $S = \text{cl}(S)$: "all points on my boundary are mine"
  $\equiv \mathbb{R}^n \setminus S$ open: "my complement owns none of my boundary"
- $\text{int}(S) \neq \emptyset \implies S$ full dimensional
- Sometimes, relative interior useful . . .

- S closed $\iff \forall\, S \supset \{\, x_i \,\} \to x \implies x \in S$
  all limit points of sequences in S are in S

- Algebra of open/closed sets:
  - $\{\, S_i \,\}$ (infinitely many) open sets $\implies \bigcup_i S_i$ open
  - S_1 and S_2 open $\implies S_1 \cap S_2$ open
  - $\{\, S_i \,\}$ (infinitely many) closed sets $\implies \bigcap_i S_i$ closed
  - S_1 and S_2 closed $\implies S_1 \cup S_2$ closed

Exercise: prove $\mathbb{R}^n$ and $\emptyset$ are both closed and open

Exercise: exhibit a set that is neither open nor closed

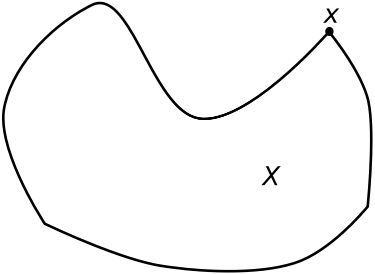
Exercise: $\{\, S_i \,\}$ (infinitely many) open sets $\implies \bigcap_i S_i$ open: true?

Exercise: $\{\, S_i \,\}$ (infinitely many) closed sets $\implies \bigcup_i S_i$ closed: true?

Outline

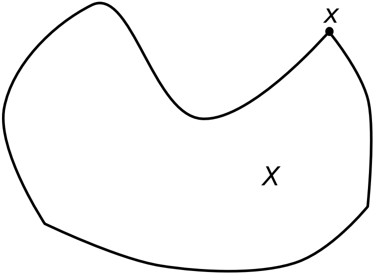
- Crucial object: $T_X(x)$ = tangent cone of X at x =

  $\{ d \in \mathbb{R}^n : \exists \{ z_i \in X \} \to x \wedge \{ t_i > 0 \} \to 0 \text{ s.t. } d = \lim_{i \to \infty} (z_i - x) / t_i \}$



- Crucial object: $T_X(x)$ = tangent cone of X at x =

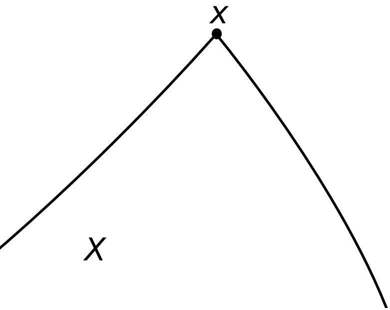
  $\{ d \in \mathbb{R}^n \, : \, \exists \{ z_i \in X \} \to x \, \wedge \, \{ t_i > 0 \} \to 0 \text{ s.t. } d = \lim_{i \to \infty} (z_i - x) / t_i \}$



- Er ... what? Simpler than it seems

- Crucial object: $T_X(x)$ = tangent cone of X at x =

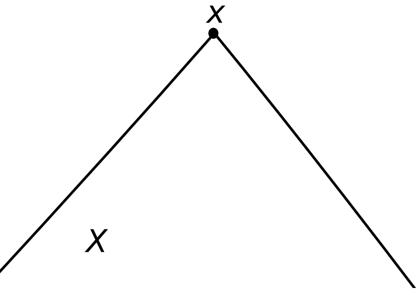
  $\{ d \in \mathbb{R}^n : \exists \{ z_i \in X \} \to x \wedge \{ t_i > 0 \} \to 0 \text{ s.t. } d = \lim_{i \to \infty} (z_i - x) / t_i \}$



- Er . . . what? Simpler than it seems
- Zoom to x

- Crucial object: $T_X(x)$ = tangent cone of X at x =

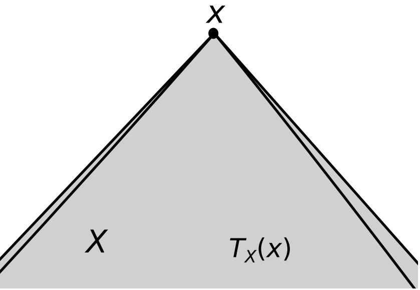
  $\{ d \in \mathbb{R}^n : \exists \{ z_i \in X \} \to x \wedge \{ t_i > 0 \} \to 0 \text{ s.t. } d = \lim_{i\to\infty}(z_i - x) / t_i \}$



- Er ... what? Simpler than it seems
- Zoom to x very closely, then

- Crucial object: $T_X(x)$ = tangent cone of X at x =

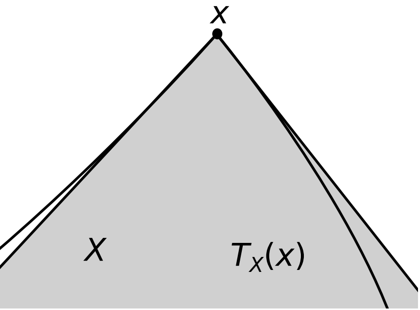
$$\{ d \in \mathbb{R}^n : \exists \{ z_i \in X \} \to x \wedge \{ t_i > 0 \} \to 0 \text{ s.t. } d = \lim_{i\to\infty} (z_i - x) / t_i \}$$



- Er . . . what? Simpler than it seems
- Zoom to x very closely, then X looks a cone:

- Crucial object: $T_X(x)$ = tangent cone of X at x =

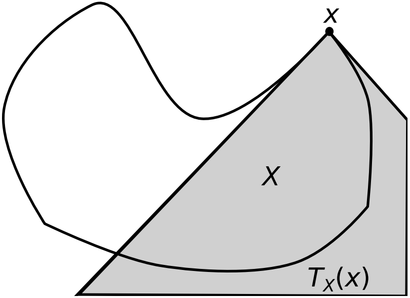
$$\{ d \in \mathbb{R}^n : \exists \{ z_i \in X \} \to x \wedge \{ t_i > 0 \} \to 0 \text{ s.t. } d = \lim_{i\to\infty} (z_i - x) / t_i \}$$



- Er . . . what? Simpler than it seems
- Zoom to x very closely, then X looks a cone: zoom out,

- Crucial object: $T_X(x)$ = tangent cone of X at x =

$$\{ d \in \mathbb{R}^n : \exists \{ z_i \in X \} \to x \wedge \{ t_i > 0 \} \to 0 \text{ s.t. } d = \lim_{i \to \infty} (z_i - x) / t_i \}$$

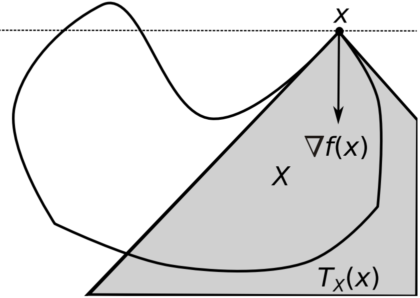


- Er . . . what? Simpler than it seems
- Zoom to x very closely, then X looks a cone: zoom out, this is $T_X(x)$

The Tangent Cone [8, § 12.2]

- Crucial object: $T_X(x)$ = tangent cone of X at x =

$$\{ d \in \mathbb{R}^n \, : \, \exists \{ z_i \in X \} \to x \, \wedge \, \{ t_i > 0 \} \to 0 \text{ s.t. } d = \lim_{i \to \infty} (z_i - x) / t_i \}$$

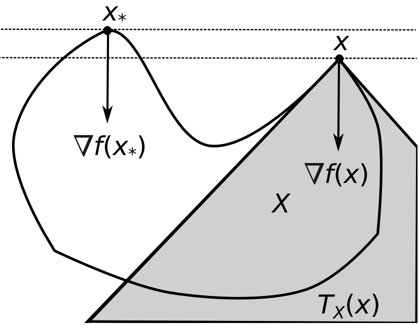


- Er ... what? Simpler than it seems
- Zoom to x very closely, then X looks a cone: zoom out, this is $T_X(x)$
- Tangent Cone Condition:
  (TCC) $\langle \nabla f(x), d \rangle \geq 0 \;\; \forall\, d \in T_X(x)$
- x local optimum $\implies$ (TCC)

Exercise: $\mathcal{C}$ cone $\equiv$ $x \in \mathcal{C} \implies \alpha x \in \mathcal{C} \; \forall \, \alpha > 0$; prove $T_X(x)$ is a cone

- Crucial object: $T_X(x)$ = tangent cone of X at x =

$\{\, d \in \mathbb{R}^n \;:\; \exists\, \{ z_i \in X \} \to x \;\wedge\; \{ t_i > 0 \} \to 0 \text{ s.t. } d = \lim_{i\to\infty} (\, z_i - x \,) \,/\, t_i \,\}$



- Er … what? Simpler than it seems
- Zoom to x very closely, then X looks a cone: zoom out, this is $T_X(x)$
- Tangent Cone Condition:

(TCC) $\quad \langle \nabla f(x), d \rangle \geq 0 \;\; \forall\, d \in T_X(x)$

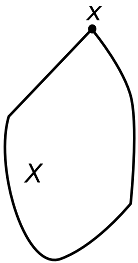
- x local optimum $\implies$ (TCC)

Exercise: $\mathcal{C}$ cone $\equiv$ $x \in \mathcal{C} \implies \alpha x \in \mathcal{C} \;\forall\, \alpha > 0$; prove $T_X(x)$ is a cone

- (TCC) $\not\Rightarrow$ local optimum (will see why), even less global optimum

- Crucial object: $T_X(x)$ = tangent cone of X at x =

$\{ d \in \mathbb{R}^n \;:\; \exists \{ z_i \in X \} \to x \;\wedge\; \{ t_i > 0 \} \to 0 \text{ s.t. } d = \lim_{i \to \infty} (z_i - x) / t_i \}$



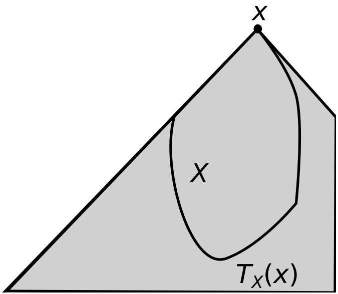
- Er ... what? Simpler than it seems
- Zoom to x very closely, then X looks a cone: zoom out, this is $T_X(x)$
- Tangent Cone Condition:
  (TCC) $\langle \nabla f(x), d \rangle \geq 0 \;\; \forall d \in T_X(x)$
- x local optimum $\Longrightarrow$ (TCC)

Exercise: $\mathcal{C}$ cone $\equiv$ $x \in \mathcal{C} \implies \alpha x \in \mathcal{C} \; \forall \alpha > 0$; prove $T_X(x)$ is a cone

- (TCC) $\not\Longrightarrow$ local optimum (will see why), even less global optimum
- Unless X

- Crucial object: $T_X(x)$ = tangent cone of X at x =

$\{ d \in \mathbb{R}^n \; : \; \exists \{ z_i \in X \} \to x \;\wedge\; \{ t_i > 0 \} \to 0 \text{ s.t. } d = \lim_{i \to \infty} (z_i - x) / t_i \}$



- Er ... what? Simpler than it seems
- Zoom to x very closely, then X looks a cone: zoom out, this is $T_X(x)$
- Tangent Cone Condition:

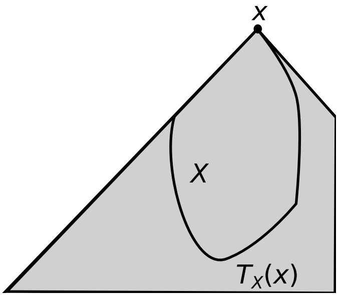
  (TCC) $\langle \nabla f(x), d \rangle \geq 0 \;\; \forall d \in T_X(x)$
- x local optimum $\Longrightarrow$ (TCC)

Exercise: $\mathcal{C}$ cone $\equiv$ $x \in \mathcal{C} \implies \alpha x \in \mathcal{C} \; \forall \alpha > 0$; prove $T_X(x)$ is a cone

- (TCC) $\not\Longrightarrow$ local optimum (will see why), even less global optimum
- Unless $X \subseteq x + T_X(x)$

- Crucial object: $T_X(x)$ = tangent cone of X at x =

  $\{ d \in \mathbb{R}^n \ : \ \exists \{ z_i \in X \} \to x \ \wedge \ \{ t_i > 0 \} \to 0 \text{ s.t. } d = \lim_{i \to \infty} (z_i - x) / t_i \}$



- Er . . . what? Simpler than it seems
- Zoom to x very closely, then X looks a cone: zoom out, this is $T_X(x)$
- Tangent Cone Condition:

  (TCC) $\langle \nabla f(x), d \rangle \geq 0 \ \forall d \in T_X(x)$
- x local optimum $\Longrightarrow$ (TCC)

Exercise: $\mathcal{C}$ cone $\equiv x \in \mathcal{C} \Longrightarrow \alpha x \in \mathcal{C} \ \forall \alpha > 0$; prove $T_X(x)$ is a cone

- (TCC) $\not\Longrightarrow$ local optimum (will see why), even less global optimum
- Unless $X \subseteq x + T_X(x) \Longleftarrow X$ convex, let's see it in details

- Prove: x local optimum $\Longrightarrow$ (TCC)
- By contradiction: x local optimum but $\exists\, d \in T_X(\, x\,)$ s.t. $\langle\, \nabla f(\, x\,)\,,\, d\,\rangle < 0$

  $d \neq 0$ and $T_X(\, x\,)$ a cone $\Longrightarrow$ w.l.o.g. $\|\, d\,\| = 1$. $d \in T_X(\, x\,) \equiv$

  $\exists\, X \supset \{\, z_i\,\} \to x\,,\ \{\, t_i\,\} \to 0$ s.t. $\lim_{i\to\infty} d - (\, z_i - x\,)\,/\,t_i = 0 \Longrightarrow$

  $\lim_{i\to\infty} t_i = \lim_{i\to\infty} \|\, z_i - x\,\|$ (t_i and $\|\, z_i - x\,\|$ "$\to 0$ at the same speed")

  First-order Taylor: $f(\, z_i\,) - f(\, x\,) = \langle\, \nabla f(\, x\,)\,,\, z_i - x\,\rangle + R(\, z_i - x\,)$

  with $\lim_{\|\, h\,\|\to 0} R(\, h\,)\,/\,\|\, h\,\| = 0$

  Crucial step: $z_i - x \approx d$ and $\langle\, \nabla f(\, x\,)\,,\, d\,\rangle < 0 \Longrightarrow \langle\, \nabla f(\, x\,)\,,\, z_i - x\,\rangle < 0$

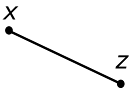
  and $\to 0$ "as fast as $z_i \to x$", while $R(\, z_i - x\,) \to 0$ "faster than $z_i \to x$"

  $\Longrightarrow$ eventually $f(\, z_i\,) - f(\, x\,) < 0$ (**check**) [a bit tedious]

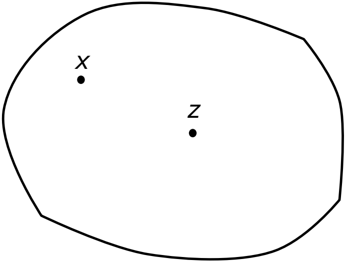
  $\{\, z_i\,\} \to x \Longrightarrow \forall$ (small) $\varepsilon > 0\ \exists\, z_i \in X \cap \mathcal{B}(\, x_*\,,\, \varepsilon\,)$ s.t. $f(\, z_i\,) < f(\, x\,)$ ↯
- $T_X(\, x\,)$ carefully defined to make the proof work

  (but as it is, the definition in unwieldy and unworkable)

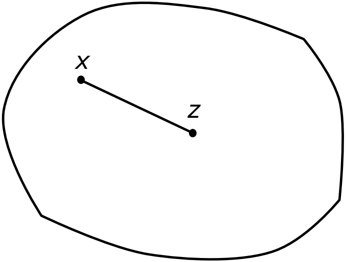
- $x, z \in \mathbb{R}^n$,



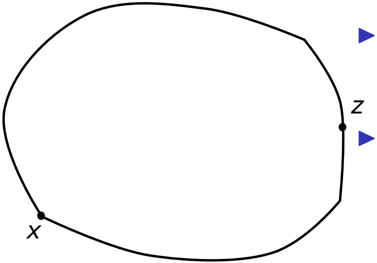
- $x\,,\, z \in \mathbb{R}^n, \{\, w = \alpha x + (1-\alpha) z \,:\, \alpha \in [\,0\,,\,1\,]\,\}$
  $= conv(\,x\,,\,z\,) =$ segment joining x and z



- $x\,,\,z \in \mathbb{R}^n, \{\, w = \alpha x + (1-\alpha)z \;:\; \alpha \in [\,0\,,\,1\,]\,\}$
  $= conv(\,x\,,\,z\,) =$ segment joining x and z
- $C \subset \mathbb{R}^n$ convex: $\forall\, x\,,\,z \in C$

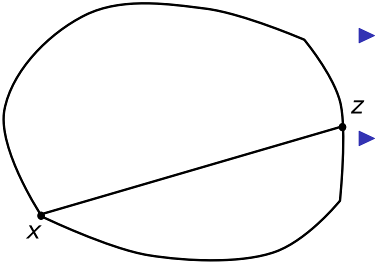


- $x\,,\,z \in \mathbb{R}^n, \{\, w = \alpha x + (1-\alpha) z \;:\; \alpha \in [\,0\,,\,1\,]\,\}$
  $= conv(\,x\,,\,z\,) =$ segment joining x and z
- $C \subset \mathbb{R}^n$ convex: $\forall\, x\,,\,z \in C \;\; conv(\,x\,,\,z\,) \subseteq C$

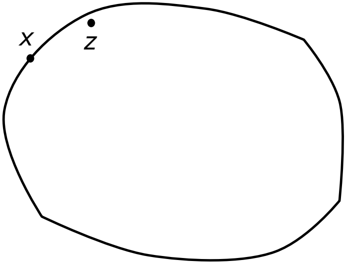


- $x\,,\,z \in \mathbb{R}^n, \{\,w = \alpha x + (1-\alpha)z\ :\ \alpha \in [\,0\,,\,1\,]\,\}$

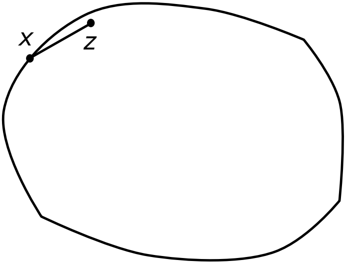
  $=$ *conv*$(\,x\,,\,z\,)$ $=$ segment joining x and z
- $C \subset \mathbb{R}^n$ convex: $\forall\,x\,,\,z \in C$ *conv*$(\,x\,,\,z\,) \subseteq C$



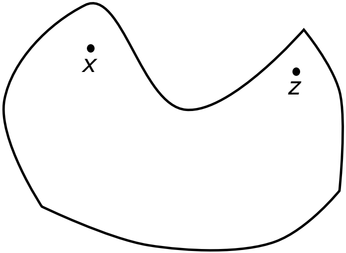
- $x\,,\,z \in \mathbb{R}^n$, $\{\, w = \alpha x + (1-\alpha) z \,:\, \alpha \in [\,0\,,\,1\,]\,\}$
  $= conv(\,x\,,\,z\,)$ = segment joining x and z
- $C \subset \mathbb{R}^n$ convex: $\forall\, x\,,\,z \in C\;\; conv(\,x\,,\,z\,) \subseteq C$



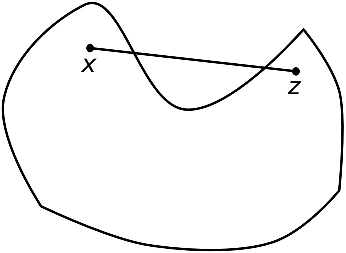
- $x\,,\,z \in \mathbb{R}^n, \{\, w = \alpha x + (1-\alpha) z \;:\; \alpha \in [\,0\,,\,1\,]\,\}$
  $= conv(\,x\,,\,z\,) =$ segment joining x and z
- $C \subset \mathbb{R}^n$ convex: $\forall\, x\,,\,z \in C \;\; conv(\,x\,,\,z\,) \subseteq C$



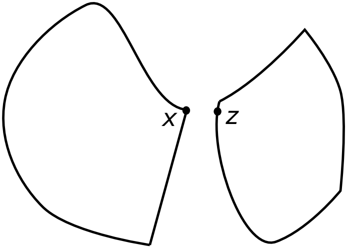
- $x\,,\,z \in \mathbb{R}^n, \{\, w = \alpha x + (1-\alpha) z \;:\; \alpha \in [\,0\,,\,1\,]\,\}$
  $= conv(\,x\,,\,z\,) =$ segment joining x and z
- $C \subset \mathbb{R}^n$ convex: $\forall\, x\,,\,z \in C \;\; conv(\,x\,,\,z\,) \subseteq C$



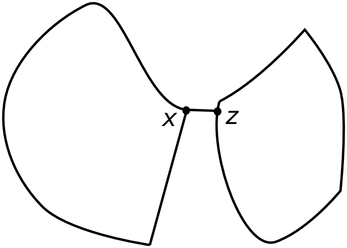
- $x\,,\,z \in \mathbb{R}^n, \{\, w = \alpha x + (1-\alpha) z \;:\; \alpha \in [\,0\,,\,1\,]\,\}$
  $= conv(\,x\,,\,z\,) =$ segment joining x and z
- $C \subset \mathbb{R}^n$ convex: $\forall\, x\,,\,z \in C$ $conv(\,x\,,\,z\,) \subseteq C$
- C nonconvex: $\exists\, x\,,\,z \in C$ s.t.



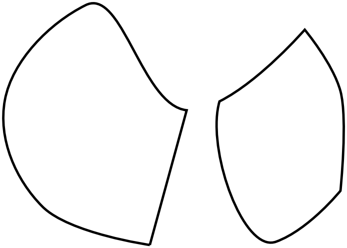
- $x\,,\,z \in \mathbb{R}^n, \{\, w = \alpha x + (1-\alpha) z \;:\; \alpha \in [\,0\,,\,1\,]\,\}$
  $= conv(\,x\,,\,z\,) =$ segment joining x and z
- $C \subset \mathbb{R}^n$ convex: $\forall\, x\,,\,z \in C \;\; conv(\,x\,,\,z\,) \subseteq C$
- C nonconvex: $\exists\, x\,,\,z \in C$ s.t. $conv(\,x\,,\,z\,) \not\subseteq C$



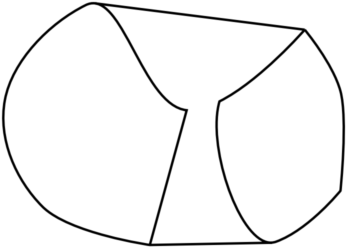
- $x\,,\,z \in \mathbb{R}^n$, $\{\, w = \alpha x + (1-\alpha) z \;:\; \alpha \in [\,0\,,\,1\,]\,\}$
  $= conv(\,x\,,\,z\,) =$ segment joining x and z
- $C \subset \mathbb{R}^n$ convex: $\forall\, x\,,\,z \in C$ $conv(\,x\,,\,z\,) \subseteq C$
- C nonconvex: $\exists\, x\,,\,z \in C$ s.t. $conv(\,x\,,\,z\,) \not\subseteq C$
- Convex is rare: non-connected



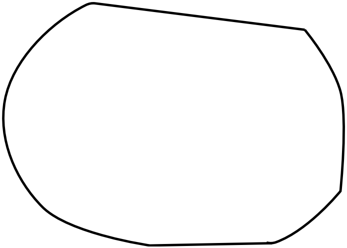
- $x\,,\,z \in \mathbb{R}^n, \{\, w = \alpha x + (1-\alpha) z \;:\; \alpha \in [\,0\,,\,1\,]\,\}$ $= conv(\,x\,,\,z\,) =$ segment joining x and z
- $C \subset \mathbb{R}^n$ convex: $\forall\, x\,,\,z \in C$ $conv(\,x\,,\,z\,) \subseteq C$
- C nonconvex: $\exists\, x\,,\,z \in C$ s.t. $conv(\,x\,,\,z\,) \not\subseteq C$
- Convex is rare: non-connected $\implies$ nonconvex



- $x\,, z \in \mathbb{R}^n, \{\, w = \alpha x + (1-\alpha)z \;:\; \alpha \in [\,0\,,\,1\,]\,\}$
  $= conv(\,x\,,\,z\,) =$ segment joining x and z
- $C \subset \mathbb{R}^n$ convex: $\forall\, x\,, z \in C\;\; conv(\,x\,,\,z\,) \subseteq C$
- C nonconvex: $\exists\, x\,, z \in C$ s.t. $conv(\,x\,,\,z\,) \not\subseteq C$
- Convex is rare: non-connected $\implies$ nonconvex
- Every nonconvex set can be



- $x, z \in \mathbb{R}^n, \{ w = \alpha x + (1-\alpha) z \,:\, \alpha \in [0, 1] \}$
  $= conv(x, z) =$ segment joining x and z
- $C \subset \mathbb{R}^n$ convex: $\forall x, z \in C \; conv(x, z) \subseteq C$
- C nonconvex: $\exists x, z \in C$ s.t. $conv(x, z) \not\subseteq C$
- Convex is rare: non-connected $\implies$ nonconvex
- Every nonconvex set can be "completed"



- $x\,,\,z \in \mathbb{R}^n$, $\{\,w = \alpha x + (1-\alpha)z \;:\; \alpha \in [\,0\,,\,1\,]\,\}$
  $= conv(\,x\,,\,z\,) =$ segment joining x and z
- $C \subset \mathbb{R}^n$ convex: $\forall\,x\,,\,z \in C\;\; conv(\,x\,,\,z\,) \subseteq C$
- C nonconvex: $\exists\,x\,,\,z \in C$ s.t. $conv(\,x\,,\,z\,) \not\subseteq C$
- Convex is rare: non-connected $\implies$ nonconvex
- Every nonconvex set can be "completed" to a convex set (convex hull):
  $conv(\,S\,) = \bigcup\{\,conv(\,x\,,\,z\,) \;:\; x\,,\,z \in S\,\} = \bigcap\{\,C \;:\; C \text{ is convex } \wedge C \supseteq S\,\}$
  $=$ "iterated convex hull of all $x\,,\,z \in S$" $=$ smallest convex set containing S
- C convex $\iff C = conv(\,C\,)$: a convex set is equal to its convex hull

Exercise: prove: $\text{epi}(\,f\,) = \{\,(\,v\,,\,x\,) \;:\; v \geq f(\,x\,)\,\}$ convex $\iff f$ convex

- f convex $\implies S(\,f\,,\,v\,)$ convex $\forall\,v$ $(\not\Longleftarrow)$ (**check**)
- How do you tell if a set is convex?

- A few sets are "obviously" convex:
  - i Convex hull of a finite set of points = convex polytope =
    $conv(\{ x_1 , \ldots , x_k \}) = \{ x = \sum_{i=1}^{k} \alpha_i x_i \; : \; \sum_{i=1}^{k} \alpha_i = 1 \; , \; \alpha_i \geq 0 \;\; \forall \, i \}$
  - ii Affine hyperplane $\mathcal{H} = \{ x \in \mathbb{R}^n \; : \; ax = b \}$ = level set of linear function
  - iii Affine subspace $\mathcal{S} = \{ x \in \mathbb{R}^n \; : \; ax \leq b \}$ = sublevel set of linear function
  - iv Ellipsoid $\mathcal{E}(Q , x , r) = \{ z \in \mathbb{R}^n \; : \; (z - x)^T Q(z - x) \leq r \}$ with $Q \succeq 0$
    = sublevel set of convex quadratic function
  - v Ball in p-norm, $p \geq 1$, $\mathcal{B}_p(x , r) = \{ z \in \mathbb{R}^n \; : \; \| z - x \|_p \leq r \}$
- Some interesting convex sets are cones:
  - i Conical hull of a finite set of directions = polyhedral cone =
    $$cone(\{ d_1 , \ldots , d_k \}) = \{ d = \sum_{i=1}^{k} \mu_i d_i \; : \; \mu_i \geq 0 \;\; \forall \, i \}$$
  - ii Lorentz (ice-cream) cone $\mathbb{L} = \left\{ x \in \mathbb{R}^n \; : \; x_n \geq \sqrt{\sum_{i=1}^{n-1} x_i^2} \right\}$
  - iii Cone of positive semidefinite matrices $\mathbb{S}_+ = \{ Q \in \mathbb{R}^{n \times n} \; : \; Q \succeq 0 \}$

Exercise: prove that $\mathbb{R}^n_+$ is a convex cone in two different ways

Exercise: provide an example of a non-convex cone

Mathematically speaking: Convexity-preserving operations [2, §. 2.2]

i $\{ C_i \}_{i \in I}$ a (possibly ∞) family of convex sets $\implies \underline{C} = \bigcap_{i \in I} C_i$ convex

ii $C_1 \,,\, \ldots \,,\, C_k$ convex $\iff C_1 \times \ldots \times C_k$ convex

iii C convex $\implies A(\, C\,) = \{\, x = Az + b \,:\, z \in C \,\}$ convex
(image under a linear mapping, e.g., scaling, translation, rotation)

iv C convex $\implies A^{-1}(\, C\,) = \{\, x \,:\, Ax + b \in C \,\}$ convex
(inverse image under a linear mapping)

v C_1 and C_2 convex, $\alpha_1 \,,\, \alpha_2 \in \mathbb{R} \implies$

$$\alpha_1 C_1 + \alpha_2 C_2 = \{\, x = \alpha_1 x_1 + \alpha_2 x_2 \,:\, x_1 \in C_1 \,,\, x_2 \in C_2 \,\} \text{ convex}$$

vi $C \subseteq \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$ convex $\implies$

- i $C(\, z\,) = \{\, x \in \mathbb{R}^m \,:\, (\, x \,,\, z\,) \in C \,\}$ convex (slice)
- ii $C^1 = \{\, x \in \mathbb{R}^m \,:\, \exists\, z \in \mathbb{R}^k \text{ s.t. } (\, x \,,\, z\,) \in C \,\}$ convex (shadow)

vii C convex $\implies$ $int(\, C\,)$ and $cl(\, C\,)$ convex

Exercise: prove i., iii. and iv. "from prime principles", then v.

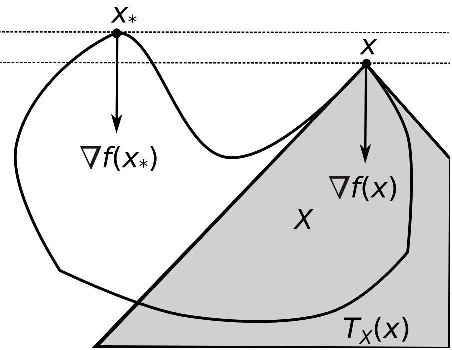
- Feasible direction d of X at x: $\exists\, \bar{\varepsilon} > 0$ s.t. $x + \bar{\varepsilon} d \in X$
- $F_X(\, x \,) =$ cone of feasible directions of X at x: $X \subseteq x + F_X(\, x \,)$ (**check**)
- X convex, $d \in F_X(\, x \,) \implies x + \varepsilon d \in X \;\; \forall \varepsilon \in [\, 0 \,, \, \bar{\varepsilon} \,]$
- X convex $\implies F_X(\, x \,) \subseteq T_X(\, x \,)$ (in fact $F_X(\, x \,) \approx T_X(\, x \,)$ "save possibly for the borders") $\implies X \subseteq x + T_X(\, x \,)$ (**check**)

Exercise: for X nonconvex, "F_X much larger than T_X": illustrate

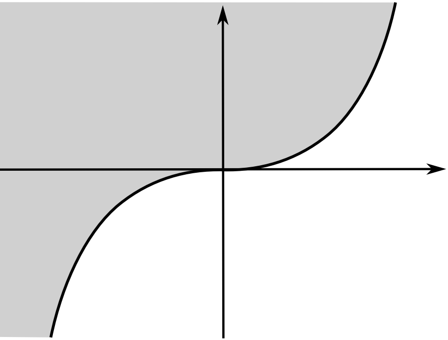
- x_* global optimum $\implies x_*$ local optimum $\implies$ (TCC) (no matter f, X)
- (P) convex $\equiv$ X convex, f convex on X: (TCC) $\implies x_*$ global optimum

Exercise: prove, discuss if $\nabla f(\, x \,)$ can be replaced by $g \in \partial f(\, x \,)$ when $f \notin C^1$

- (TCC) sufficient in the convex case, always necessary

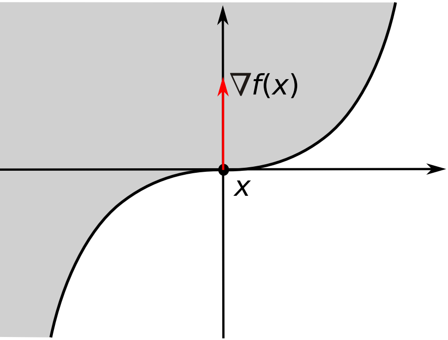


- Obvious for global: x local minimum but $\exists$ better one somewhere else



- Obvious for global: x local minimum but $\exists$ better one somewhere else
- Less obvious for local:

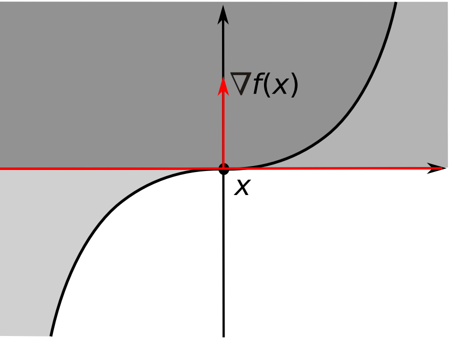
  $\min\{ x_2 \ : \ x_2 \geq x_1^3 \}$



- Obvious for global: x local minimum but $\exists$ better one somewhere else
- Less obvious for local:

  $\min\{\, x_2 \,:\, x_2 \geq x_1^3 \,\}$

  $x = [\,0\,,\,0\,]$, $\nabla f(\,x\,) = [\,0\,,\,1\,]$,



- Obvious for global: x local minimum but $\exists$ better one somewhere else
- Less obvious for local:
  $\min\{\, x_2 \,:\, x_2 \geq x_1^3 \,\}$
  $x = [\,0\,,\,0\,]$, $\nabla f(\,x\,) = [\,0\,,\,1\,]$,
  $T_X(\,x\,) = \{\,[\,x_1\,,\,x_2\,]\,:\,x_2 \geq 0\,\}$
- (TCC) holds but x not minimum

- $\exists$ better x' arbitrarily close to x, but not along a straight line (and derivatives "only look at straight lines")
- Clearly due to nonconvexity: x a "saddle point of ∂X"

Say "hi" to the stationary point of constrained case

- All in all: (TCC) $\equiv$ "stationary point of constrained case"

- All in all: (TCC) $\equiv$ "stationary point of constrained case"
- Clearly $\nabla f(x) = 0 \implies \langle \nabla f(x), d \rangle = 0 \;\forall\, d \in \mathbb{R}^n \implies$ (TCC)
  a (feasible) local minimum is a local optimum regardless X

- All in all: (TCC) $\equiv$ "stationary point of constrained case"
- Clearly $\nabla f(x) = 0 \implies \langle \nabla f(x), d \rangle = 0 \;\forall d \in \mathbb{R}^n \implies$ (TCC)
  a (feasible) local minimum is a local optimum regardless X
- Conversely, $x \in int(X) \implies T_X(x) = F_X(x) = \mathbb{R}^n$ (**check**), hence

  (TCC) $\equiv \langle \nabla f(x), d \rangle \geq 0 \;\forall d \in \mathbb{R}^n \equiv \nabla f(x) = 0$ (**check**)

  the only way for $x \in int(X)$ to be a local optimum is to be a local minimum
- In fact, f, X convex $\implies$ (TCC) $\equiv \nexists$ feasible descent directions:
  $X = \mathbb{R}^n \implies$ every direction is feasible, hence only descent matters

- All in all: (TCC) $\equiv$ “stationary point of constrained case”
- Clearly $\nabla f(x) = 0 \implies \langle \nabla f(x), d \rangle = 0 \;\forall\, d \in \mathbb{R}^n \implies$ (TCC)
  a (feasible) local minimum is a local optimum regardless X
- Conversely, $x \in int(X) \implies T_X(x) = F_X(x) = \mathbb{R}^n$ (**check**), hence
  $$\text{(TCC)} \equiv \langle \nabla f(x), d \rangle \geq 0 \;\forall\, d \in \mathbb{R}^n \equiv \nabla f(x) = 0 \quad \text{(\textbf{check})}$$
  the only way for $x \in int(X)$ to be a local optimum is to be a local minimum
- In fact, f, X convex $\implies$ (TCC) $\equiv$ $\nexists$ feasible descent directions:
  $X = \mathbb{R}^n \implies$ every direction is feasible, hence only descent matters
- “x satisfies (TCC)” direct constrained generalisation of “x stationary point”
- Necessary, not sufficient, but the only one you can reasonably check

- All in all: (TCC) $\equiv$ "stationary point of constrained case"
- Clearly $\nabla f(x) = 0 \implies \langle \nabla f(x), d \rangle = 0 \;\forall d \in \mathbb{R}^n \implies$ (TCC)
  a (feasible) local minimum is a local optimum regardless X
- Conversely, $x \in int(X) \implies T_X(x) = F_X(x) = \mathbb{R}^n$ (**check**), hence
  $$\text{(TCC)} \equiv \langle \nabla f(x), d \rangle \geq 0 \;\forall d \in \mathbb{R}^n \equiv \nabla f(x) = 0 \quad (\textbf{check})$$
  the only way for $x \in int(X)$ to be a local optimum is to be a local minimum
- In fact, f, X convex $\implies$ (TCC) $\equiv \nexists$ feasible descent directions:
  $X = \mathbb{R}^n \implies$ every direction is feasible, hence only descent matters
- "x satisfies (TCC)" direct constrained generalisation of "x stationary point"
- Necessary, not sufficient, but the only one you can reasonably check
- But how to compute $T_X(x)$ / test (TCC) in practice? Prove something $\nexists$??
- How to characterize T_X depends on how you characterize X

Outline

- The most used way to describe a set is via (more than) one function(s)
- The obvious way: inequality constraint $f(x) \leq \delta \equiv$ sublevel set $S(f, \delta)$
  equality constraint $f(x) = \delta \equiv$ level set $L(f, \delta)$
- For convenience "δ hidden in f" $\implies f(x) \leq 0$, $f(x) = 0$
- What if one rather wants $f(x) \geq 0$? Simply $-f(x) \leq 0$
- Usually multiple constraints: "$f_1(x) \leq 0$, $f_2(x) \leq 0$" $\equiv$ logical conjunction ("first condition and second condition") $\equiv$ intersection of (sub)level sets
- (One of the) standard form(s) of constrained nonlinear optimization:

$$X = \{ x \in \mathbb{R}^n : g_i(x) \leq 0 \;\; i \in \mathcal{I} , h_j(x) = 0 \;\; j \in \mathcal{J} \}$$

  $\mathcal{I}$ = set of inequality constraints, $\mathcal{J}$ = set of equality constraints
- Compact version via vector-valued functions

  $G(x) = [\, g_i(x) \,]_{i \in \mathcal{I}} : \mathbb{R}^n \to \mathbb{R}^{\#\mathcal{I}}$, $H(x) = [\, h_i(x) \,]_{i \in \mathcal{J}} : \mathbb{R}^n \to \mathbb{R}^{\#\mathcal{J}}$

$$X = \{ x \in \mathbb{R}^n : G(x) \leq 0 , H(x) = 0 \}$$

- Very important concept: there are many different ways to express the same X
- Often choosing the right formulation crucial for being able to solve a problem
- We will not see this here, but a few trivial observations useful
- Could always assume $\#\mathcal{J} = 0$ (no equality constraints)

  $$h_j(x) = 0 \;\equiv\; h_j(x) \leq 0 \;,\; -h_j(x) \leq 0$$

  (one equality constraint is equivalent to two "opposite" inequalities)
- Could always assume $\#\mathcal{I} = 1$ (one single inequality constraint)

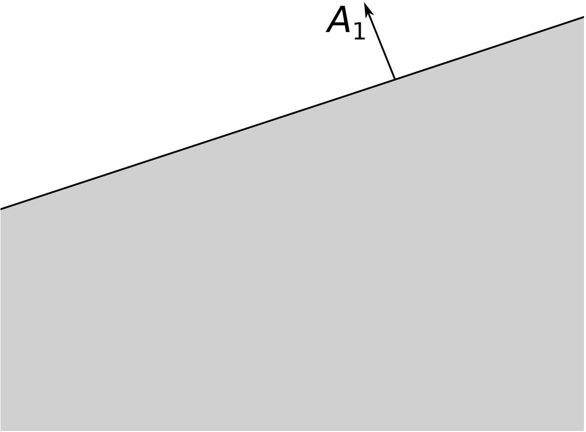
  $$G(x) \leq 0 \;\equiv\; \max\{\, g_i(x) \,:\, i \in \mathcal{I} \,\} = g(x) \leq 0$$

- Useful to simplify notation, but almost never for implementation: exploit the structure of X / the constraints when is there
- Reformulations can be bad: $\max\{\, g_1 , g_2 \,\} \notin C^1$ even if $g_1 \in C^1$, $g_2 \in C^1$

- Convexity of X important, how can I be sure of it?
- Sublevel sets of convex functions are convex
  $\Longrightarrow g_i(x) \leq 0$ with g_i convex "good"
- $g_i(x) \geq 0$ not convex if g_i is, typically "badly so" (reverse convex)
- $g_i(x) \geq 0$ convex if g_i concave, but $g_i(x) \leq 0$ then is not
- As a great man said: "convex optimization is a one-sided world"
- $g_i(x) = 0$ convex only if $g_i(x) \leq 0$ convex and $g_i(x) \geq 0$ convex
  $\equiv$ g_i is both convex and concave $\equiv$ g_i is linear (affine)
- Want a convex X? All equality constraints must be linear (affine)
- Linear constraints very important, let's give them a very good look

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$

- $g_i(x) = \langle A_i, x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i, \cdot)$

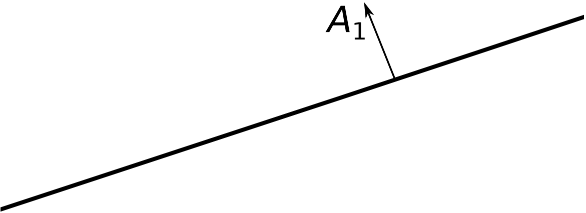


$\langle A_i, x \rangle + b_i \leq 0$

half-space (half-plane)

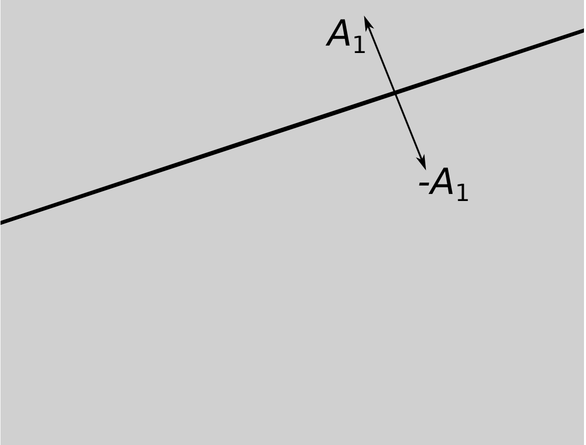
full-dimensional $\equiv$ interior $\exists$

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



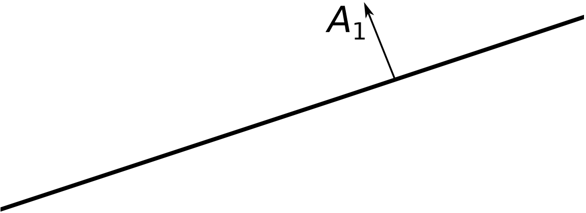
$\langle A_i , x \rangle + b_i = 0 \equiv$ hyperplane (plane, line)

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



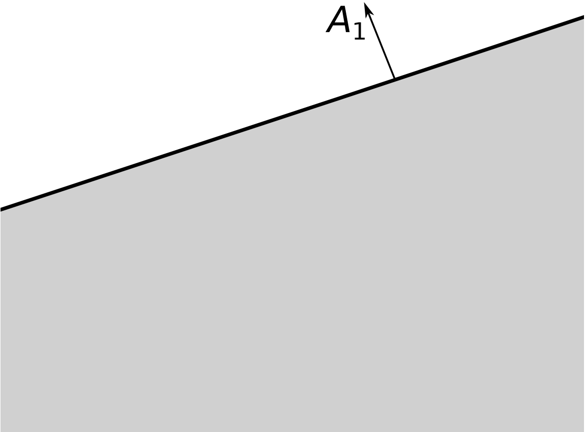
$\langle A_i , x \rangle + b_i = 0 \;\equiv$ hyperplane (plane, line) $\equiv \; \langle A_i , x \rangle + b_i \leq 0 \;,\; \langle -A_i , x \rangle + b_i \leq 0$

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



$\langle A_i , x \rangle + b_i = 0 \;\equiv$
hyperplane (plane, line)
not full-dimensional:
dimension $= n - 1$
interior $\nexists$

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



$\langle A_i , x \rangle + b_i = 0 \;\equiv$

hyperplane (plane, line)

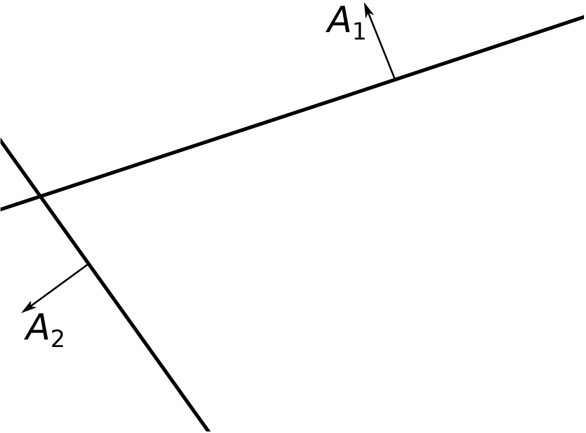
not full-dimensional:

dimension $= n - 1$

interior $\nexists$

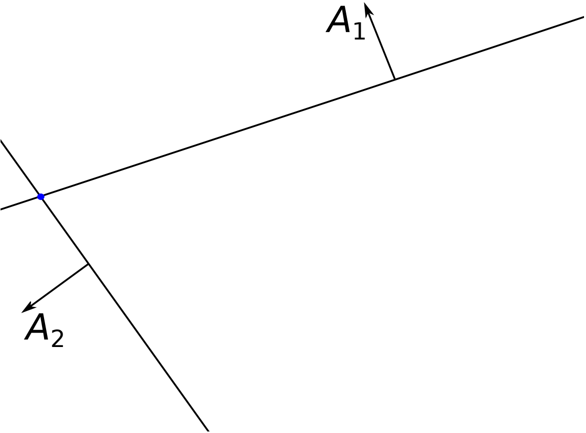
boundary of half-space

- $g_i(x) = \langle A_i, x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i, \cdot)$



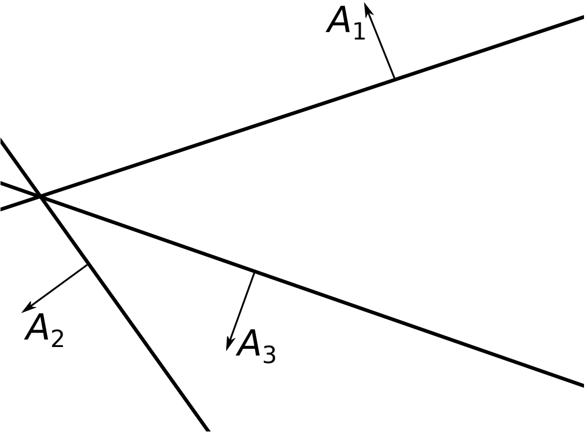
k equality constraints

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



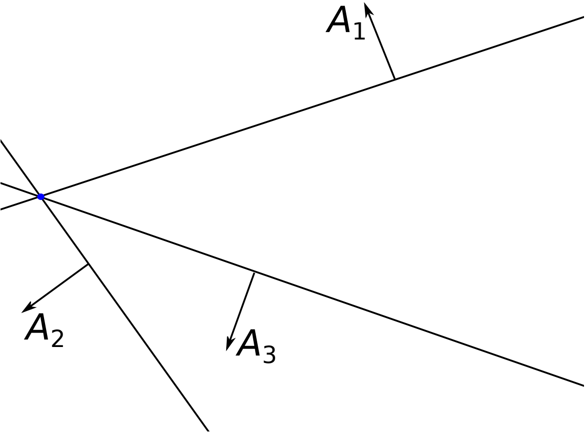
k equality constraints
(affine) subspace
dimension $\leq n - k$
$\implies k \leq n$

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



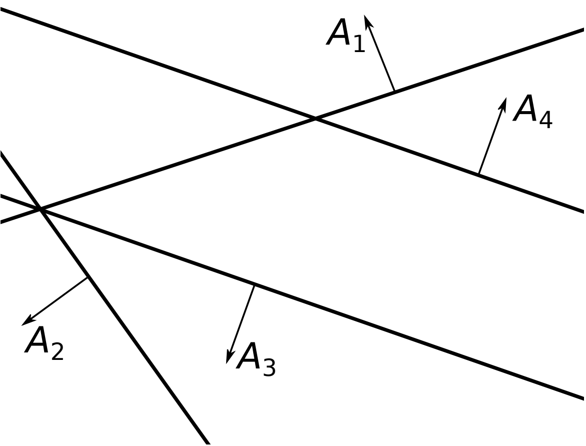
k equality constraints
(affine) subspace
dimension $\leq n - k$
$\Longrightarrow k \leq n$
unless linearly dependent

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



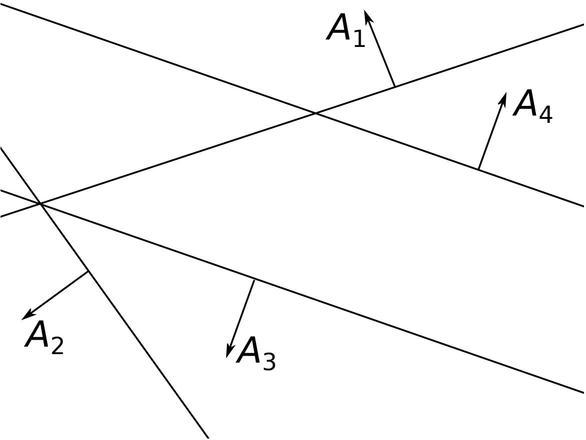
k equality constraints
(affine) subspace
dimension $\leq n - k$
$\Longrightarrow k \leq n$
unless linearly dependent

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



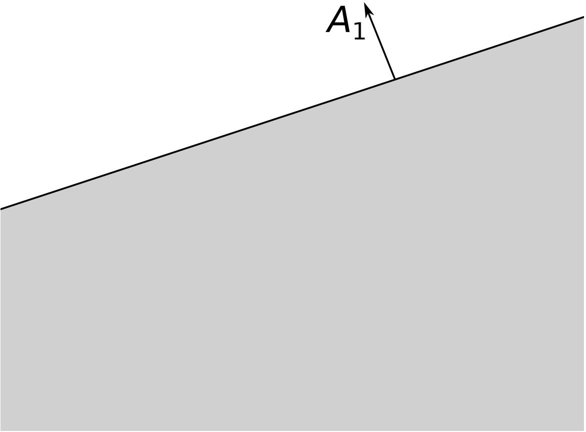
k equality constraints
(affine) subspace
dimension $\leq n - k$
$\Longrightarrow k \leq n$
unless linearly dependent

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



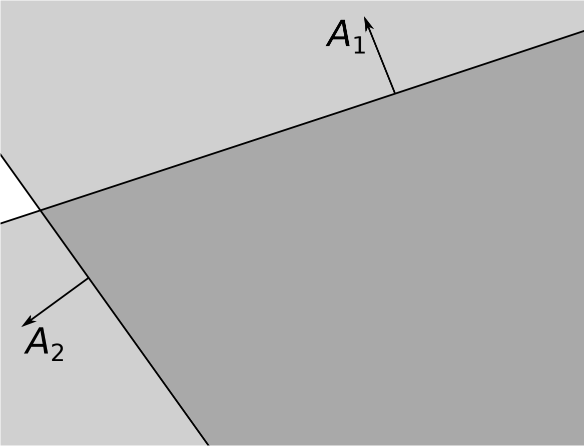
k equality constraints
(affine) subspace
dimension $\leq n - k$
$\Longrightarrow k \leq n$
unless linearly dependent
which can make it empty

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



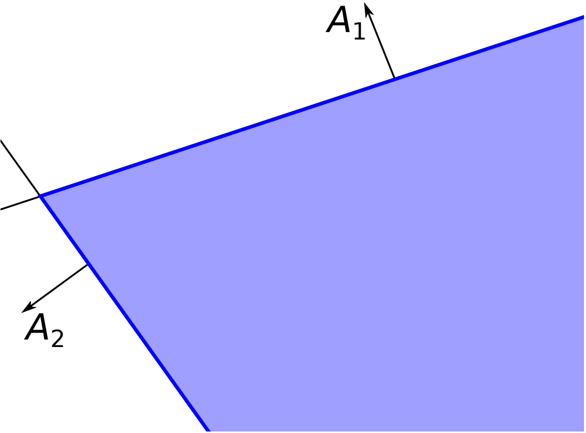
k inequality constraints

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



k inequality constraints

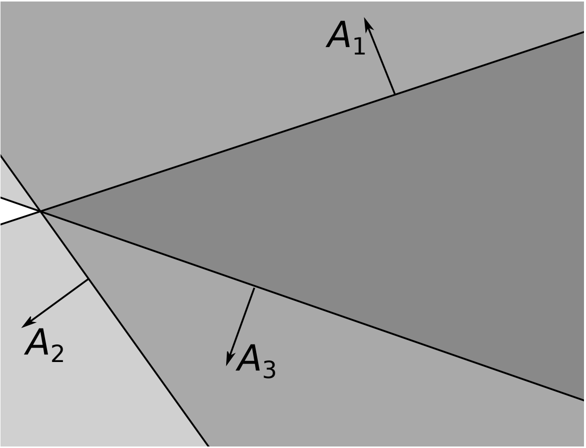
- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



k inequality constraints

polyhedral cone

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i, \cdot)$



k inequality constraints

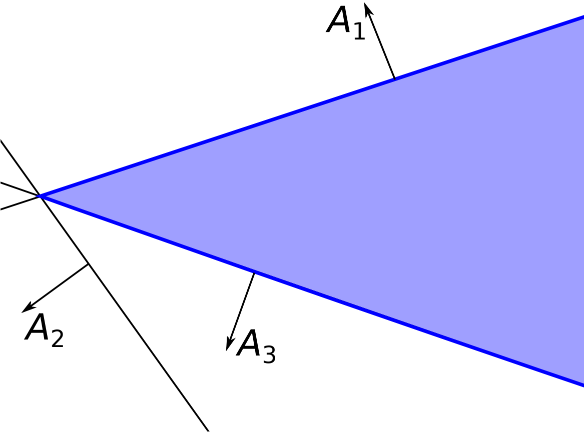
polyhedral cone

until $k \leq n$

unless linearly dependent

and having the same solution

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



k inequality constraints

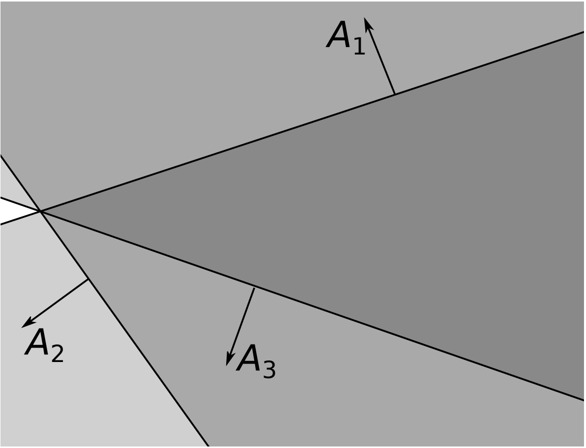
polyhedral cone

until $k \leq n$

unless linearly dependent

and having the same solution

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



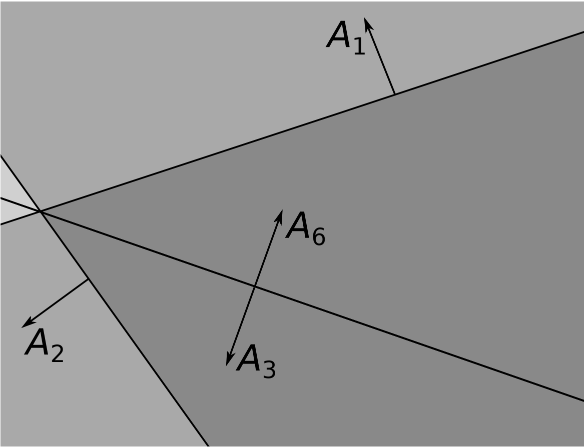
k inequality constraints

polyhedral cone

full dimensional

interior $\exists$

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



k inequality constraints

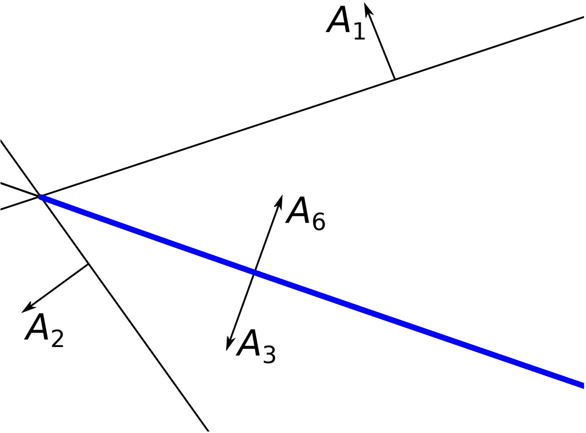
polyhedral cone

full dimensional

interior $\exists$

unless implicit equalities

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



k inequality constraints

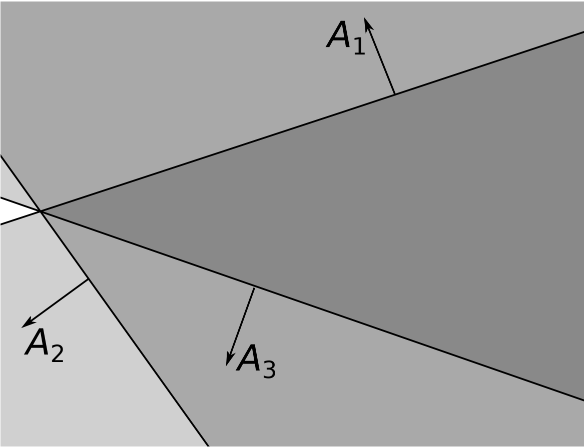
polyhedral cone

full dimensional

interior $\exists$

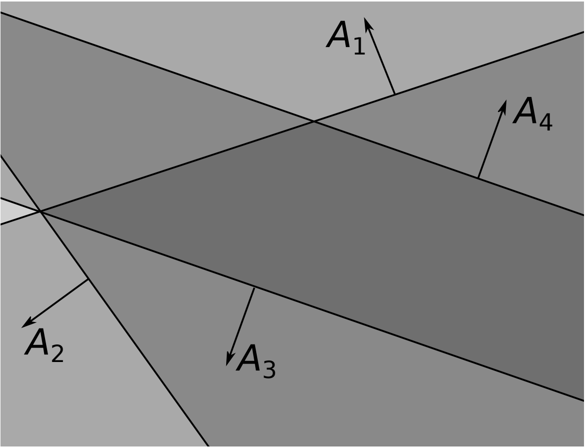
unless implicit equalities

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



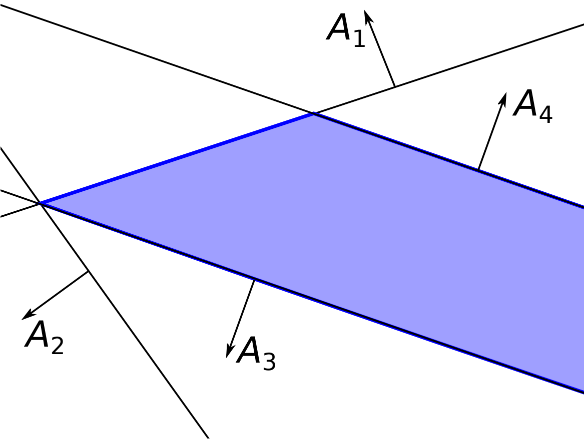
$k > n$ inequality constraints
(and not dependent "right")
convex polyhedron

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



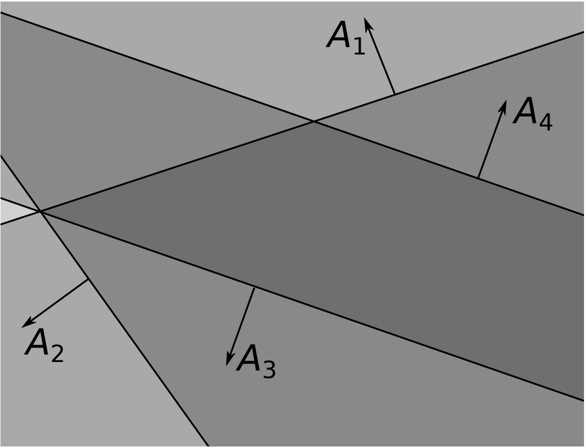
$k > n$ inequality constraints
(and not dependent "right")
convex polyhedron

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



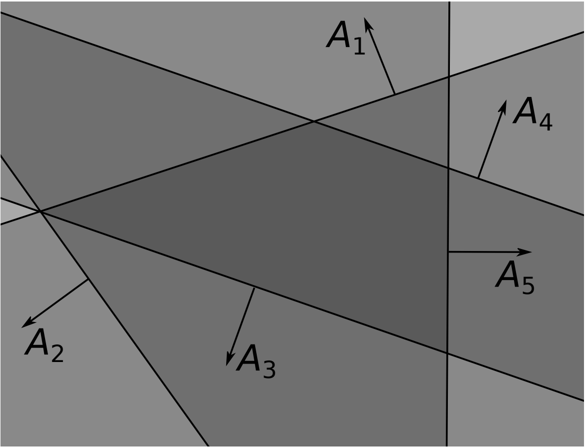
$k > n$ inequality constraints
(and not dependent "right")
convex polyhedron
unbounded

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



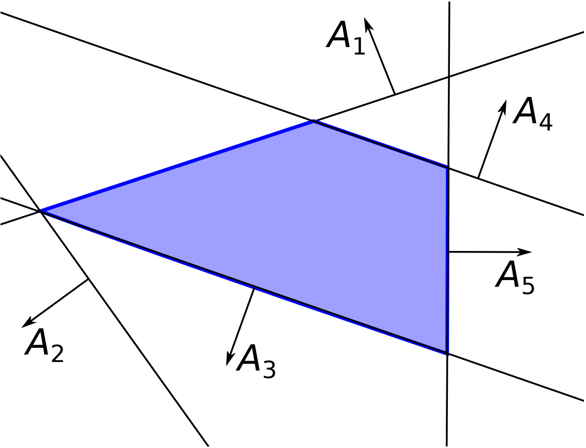
$k > n$ inequality constraints
(and not dependent "right")
convex polyhedron

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i, \cdot)$



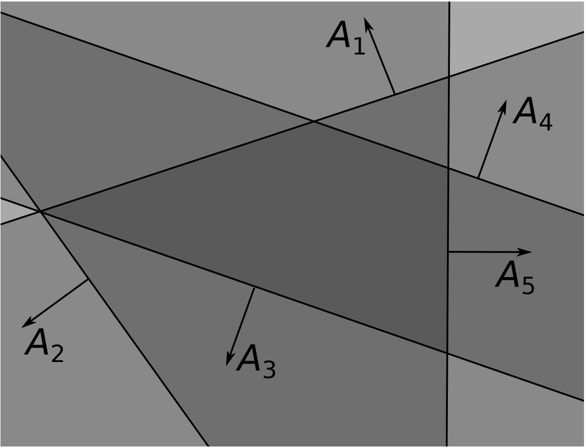
$k > n$ inequality constraints
(and not dependent "right")
convex polyhedron

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



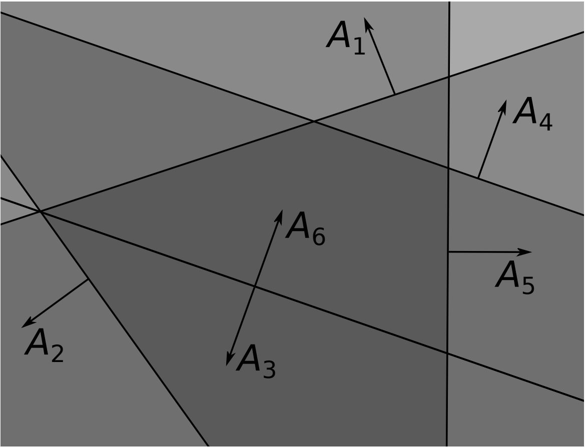
$k > n$ inequality constraints
(and not dependent "right")
convex polyhedron
bounded

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



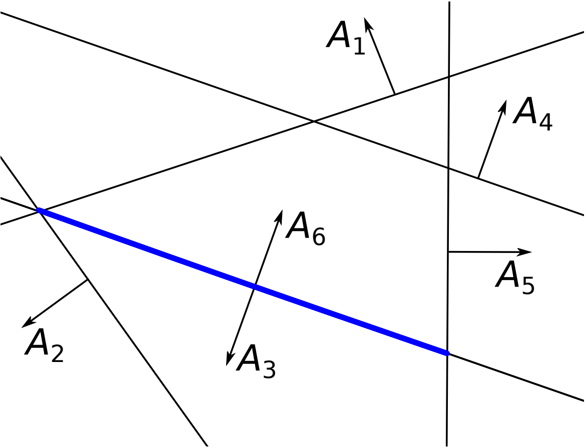
$k > n$ inequality constraints
(and not dependent "right")
convex polyhedron
implicit (or explicit)
equalities $\Longrightarrow$

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



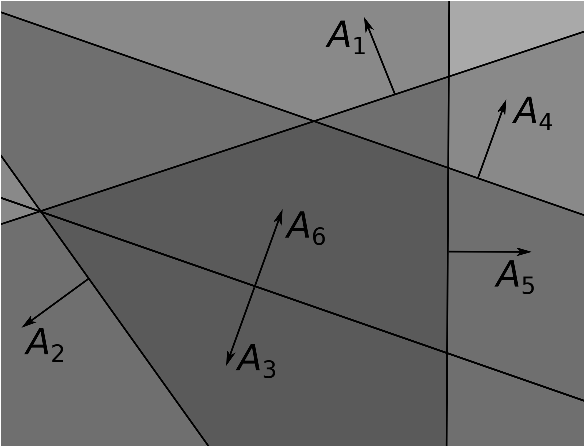
$k > n$ inequality constraints
(and not dependent "right")
convex polyhedron
implicit (or explicit)
equalities $\Longrightarrow$

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



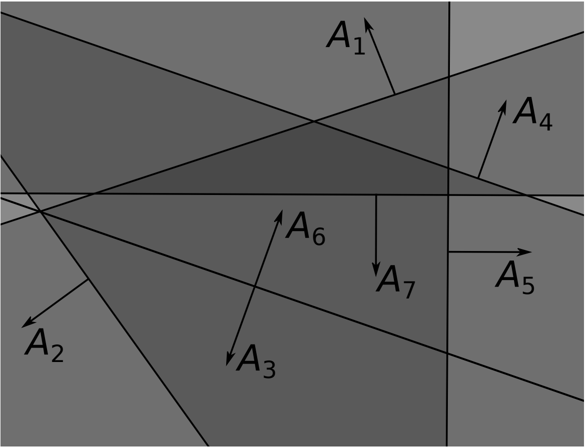
$k > n$ inequality constraints
(and not dependent "right")
convex polyhedron
implicit (or explicit)
equalities $\Longrightarrow$
non full-dimensional
interior $\nexists$

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i, \cdot)$



$k > n$ inequality constraints
(and not dependent "right")
convex polyhedron

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



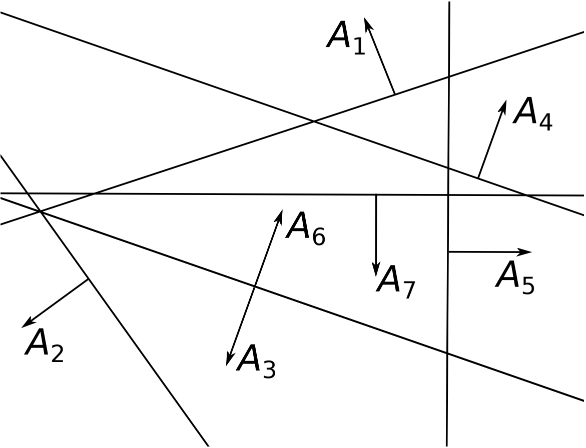
$k > n$ inequality constraints

(and not dependent "right")

convex polyhedron

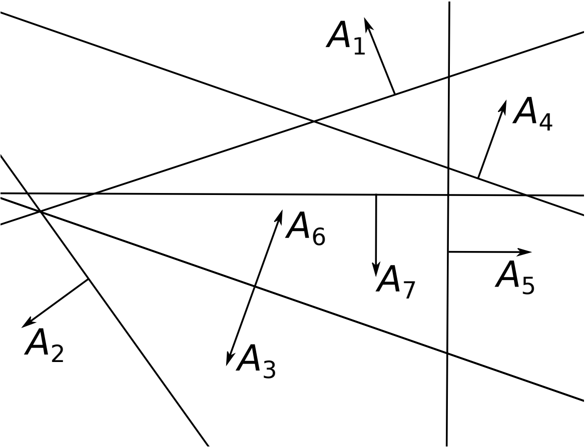
"too many constraints"

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



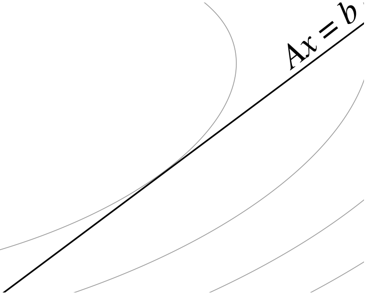
$k > n$ inequality constraints
(and not dependent "right")
convex polyhedron
"too many constraints"
$\Longrightarrow$ empty

- $g_i(x) = \langle A_i , x \rangle + b_i$ linear (affine), $\nabla g_i(x) = A_i \perp S(g_i , \cdot)$



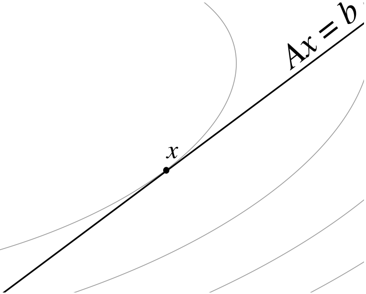
- Polyhedra are easy: find if solution $\exists \; \equiv$ optimize a linear function upon is $\mathcal{P}$
- If they have "few" constraint, or an efficient separation oracle

- Simple case, linear equality constraints: $\quad (P) \; \min\{\, f(x) \,:\, Ax = b \,\}$
- $x \in X \;\equiv\; x \in \partial X$, plus "$\partial X$ looks the same everywhere"



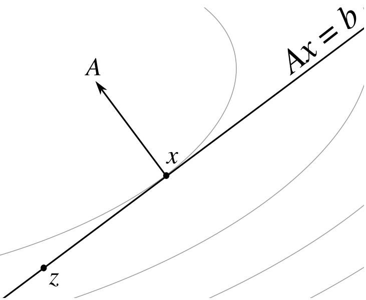
- $S(f, \cdot)$ and $X = \{\, x \in \mathbb{R}^n \,:\, Ax = b \,\}$

- Simple case, linear equality constraints: $(P)\ \min\{\, f(\,x\,) \;:\; Ax = b\,\}$
- $x \in X \;\equiv\; x \in \partial X$, plus "$\partial X$ looks the same everywhere"



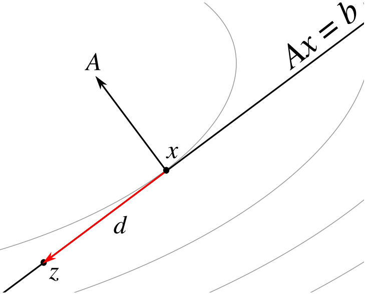
- $S(\,f\,,\,\cdot\,)$ and $X = \{\, x \in \mathbb{R}^n \;:\; Ax = b\,\}$
- optimum touches inner level set

- Simple case, linear equality constraints: $(P)\ \min\{\, f(\,x\,) \;:\; Ax = b \,\}$
- $x \in X \;\equiv\; x \in \partial X$, plus "$\partial X$ looks the same everywhere"



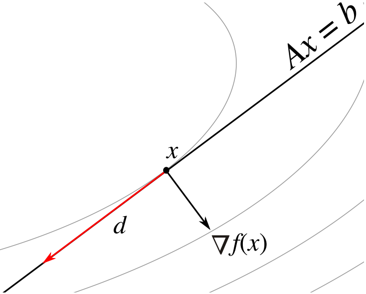
- $S(\,f\,,\,\cdot\,)$ and $X = \{\, x \in \mathbb{R}^n \;:\; Ax = b \,\}$
- optimum touches inner level set
- $F_X(\,x\,) = \{\, d \in \mathbb{R}^n \;:\; Ad = 0 \,\}\ \forall\, x$:
  for any $z \in X$,

- Simple case, linear equality constraints: $(P)\ \min\{\, f(x) \,:\, Ax = b \,\}$
- $x \in X \;\equiv\; x \in \partial X$, plus "$\partial X$ looks the same everywhere"



- $S(\, f \,, \cdot\,)$ and $X = \{\, x \in \mathbb{R}^n \,:\, Ax = b \,\}$
- optimum touches inner level set
- $F_X(\, x \,) = \{\, d \in \mathbb{R}^n \,:\, Ad = 0 \,\}\ \forall\, x$:
  for any $z \in X$, necessarily $z - x = d \perp A$

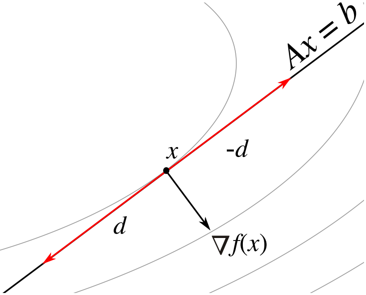
- Simple case, linear equality constraints: $(P)\ \min\{\, f(x) \,:\, Ax = b \,\}$
- $x \in X \ \equiv\ x \in \partial X$, plus "$\partial X$ looks the same everywhere"



- $S(f, \cdot)$ and $X = \{\, x \in \mathbb{R}^n \,:\, Ax = b \,\}$
- optimum touches inner level set
- $F_X(x) = \{\, d \in \mathbb{R}^n \,:\, Ad = 0 \,\}\ \forall\, x$:
  for any $z \in X$, necessarily $z - x = d \perp A$
- (TCC) $\equiv\ \langle \nabla f(x), d \rangle = 0\ \forall d \in F$, as
  $d \in F \implies$

Linear equality constraints [7, §11.3]

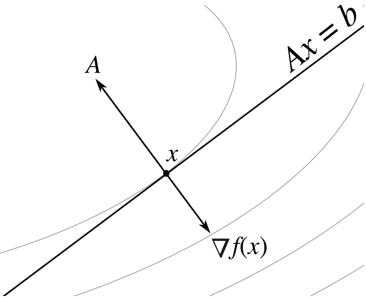
- Simple case, linear equality constraints: $(P)\ \min\{\, f(x) \,:\, Ax = b \,\}$
- $x \in X \;\equiv\; x \in \partial X$, plus "$\partial X$ looks the same everywhere"



- $S(f, \cdot)$ and $X = \{\, x \in \mathbb{R}^n \,:\, Ax = b \,\}$
- optimum touches inner level set
- $F_X(x) = \{\, d \in \mathbb{R}^n \,:\, Ad = 0 \,\}\ \forall\, x$:
  for any $z \in X$, necessarily $z - x = d \perp A$
- (TCC) $\equiv \langle \nabla f(x), d \rangle = 0\ \forall d \in F$, as
  $d \in F \implies -d \in F$:
  $d \perp A$ and $\nabla f(x) \perp d \implies$

Linear equality constraints [7, §11.3]

- Simple case, linear equality constraints: $(P)\ \min\{ f(x) : Ax = b \}$
- $x \in X \equiv x \in \partial X$, plus "$\partial X$ looks the same everywhere"



- $S(f, \cdot)$ and $X = \{ x \in \mathbb{R}^n : Ax = b \}$
- optimum touches inner level set
- $F_X(x) = \{ d \in \mathbb{R}^n : Ad = 0 \}\ \forall x$:
  for any $z \in X$, necessarily $z - x = d \perp A$
- (TCC) $\equiv \langle \nabla f(x), d \rangle = 0\ \forall d \in F$, as
  $d \in F \implies -d \in F$:
  $d \perp A$ and $\nabla f(x) \perp d \implies \nabla f(x) \parallel A$
- "$\nabla f(x) \parallel A$" $\equiv \nabla f(x) \in \mathit{range}(A) \equiv \exists\, \mu \in \mathbb{R}^m$ s.t. $\nabla f(x) = \mu A$
- "Poorman's KKT conditions": $Ax = b \ \wedge\ \exists\, \mu \in \mathbb{R}^m$ s.t. $\nabla f(x) = \mu A$
- μ first example of dual variables: to prove x optimal you have to find μ
- f convex $\implies$ (P-KKT) sufficient for global optimality (**check**)

- $A \in \mathbb{R}^{m \times n}$ with $\mathrm{rank}(A) = m < n \equiv$ rows of A linearly independent

  $\implies A = [\, A_B \,,\, A_N \,]$ with $det(A_B) \neq 0$

  $\implies x = [\, x_B \,,\, x_N \,]$ so that $Ax = b \equiv x_B = A_B^{-1}(\, b - A_N x_N \,)$

  "m linear constraints kill m degrees of freedom $\equiv$ m variables"

Exercise: why "rows of A linearly independent" makes sense? discuss

- $(P) \equiv$ reduced problem $\quad (R) \;\; \min\{\, r(x_N) = f(\, Dx_N + d \,) \,:\, x_N \in \mathbb{R}^{n-m} \,\}$

$$D = \begin{bmatrix} -A_B^{-1} A_N \\ I \end{bmatrix} \quad , \quad d = \begin{bmatrix} A_B^{-1} b \\ 0 \end{bmatrix} \quad , \quad AD = 0 \quad (\textbf{check})$$

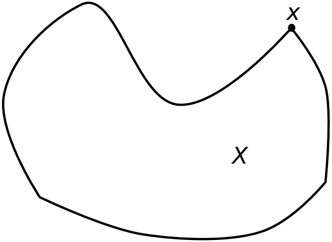
- $\nabla r(\, x_N \,) = D^T \nabla f(\, Dx_N + d \,)$, x_N^* optimal for $(R) \equiv \nabla r(\, x_N^* \,) = 0$
- $AD = 0 \implies \forall \mu \in \mathbb{R}^m,\, z^T = \mu A \implies z^T D = 0 \equiv D^T z = 0$
- $x_N^* \implies x^* = [\, x_B^* \,,\, x_N^* \,] = [\, A_B^{-1}(\, b - A_N x_N^* \,) \,,\, x_N^* \,]$ always feasible

  $\exists\, \mu \in \mathbb{R}^m$ s.t. $\mu A = \nabla f(\, x^* \,) \implies r(\, x_N^* \,) = D^T \nabla f(\, x^* \,) = 0$

  i.e., $\nabla r(\, x_N^* \,) = 0 \equiv$ (P-KKT) $\quad Ax = b \;\wedge\; \exists\, \mu \in \mathbb{R}^m$ s.t. $\mu A = \nabla f(\, x \,)$
- Can be a feasible algorithmic strategy

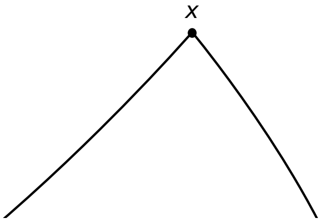
Exercise: solve $\min\{\, 2x^2 + w^2 + z^2 \,:\, x + z = 1 \,,\, x + w - z = 2 \,\}$ via (R)

- $T_X(x) = T_{X \cap \mathcal{B}(x, \varepsilon)}(x)$: only what happens as $x_i \to x$ matters
  $\Longrightarrow$ if $x \notin \partial S(g_i, 0)$ then constraint $g_i(\cdot) \leq 0$ has no impact on $T_X(x)$
- $g_i \in C^0$: $x \in \partial S(g_i, 0) \Longrightarrow g_i(x) = 0$, although $\not\Longleftarrow$ (**check**)
- Active constraints at $x \in X$: $\mathcal{A}(x) = \{ i \in \mathcal{I} : g_i(x) = 0 \} \subseteq \mathcal{I}$
  "easy proxy" of $x \in \partial S(g_i, 0) \equiv$ constraint $g_i(\cdot) \leq 0$ impacts $T_X(x)$
- First-order feasible direction cone at $x \in X$: $D_X(x) =$
  $\{ d \in \mathbb{R}^n : \langle \nabla g_i(x), d \rangle \leq 0 \;\; i \in \mathcal{A}(x), \langle \nabla h_j(x), d \rangle = 0 \;\; j \in \mathcal{J} \} =$
  $\{ d \in \mathbb{R}^n : (JG_{\mathcal{A}(x)}(x))d \leq 0, (JH(x))d = 0 \}$

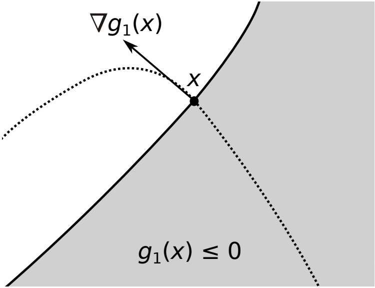


- $T_X(x) = T_{X \cap \mathcal{B}(x, \varepsilon)}(x)$: only what happens as $x_i \to x$ matters
  $\Longrightarrow$ if $x \notin \partial S(g_i, 0)$ then constraint $g_i(\cdot) \leq 0$ has no impact on $T_X(x)$
- $g_i \in C^0$: $x \in \partial S(g_i, 0) \Longrightarrow g_i(x) = 0$, although $\not\Longleftarrow$ (**check**)
- Active constraints at $x \in X$: $\mathcal{A}(x) = \{ i \in \mathcal{I} : g_i(x) = 0 \} \subseteq \mathcal{I}$
  "easy proxy" of $x \in \partial S(g_i, 0) \equiv$ constraint $g_i(\cdot) \leq 0$ impacts $T_X(x)$
- First-order feasible direction cone at $x \in X$: $D_X(x) =$
  $\{ d \in \mathbb{R}^n : \langle \nabla g_i(x), d \rangle \leq 0 \;\; i \in \mathcal{A}(x), \langle \nabla h_j(x), d \rangle = 0 \;\; j \in \mathcal{J} \} =$
  $\{ d \in \mathbb{R}^n : (JG_{\mathcal{A}(x)}(x))d \leq 0, (JH(x))d = 0 \}$

- $\mathcal{A}(x) \equiv$ zoom very close to x

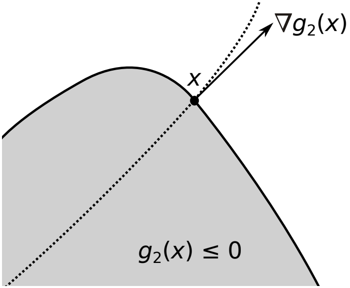


- $T_X(x) = T_{X \cap \mathcal{B}(x,\varepsilon)}(x)$: only what happens as $x_i \to x$ matters
  $\Longrightarrow$ if $x \notin \partial S(g_i, 0)$ then constraint $g_i(\cdot) \leq 0$ has no impact on $T_X(x)$
- $g_i \in C^0$: $x \in \partial S(g_i, 0) \Longrightarrow g_i(x) = 0$, although $\not\Longleftarrow$ (**check**)
- Active constraints at $x \in X$: $\mathcal{A}(x) = \{ i \in \mathcal{I} : g_i(x) = 0 \} \subseteq \mathcal{I}$
  "easy proxy" of $x \in \partial S(g_i, 0) \equiv$ constraint $g_i(\cdot) \leq 0$ impacts $T_X(x)$
- First-order feasible direction cone at $x \in X$: $D_X(x) =$
  $\{ d \in \mathbb{R}^n : \langle \nabla g_i(x), d \rangle \leq 0 \;\; i \in \mathcal{A}(x), \langle \nabla h_j(x), d \rangle = 0 \;\; j \in \mathcal{J} \} =$
  $\{ d \in \mathbb{R}^n : (JG_{\mathcal{A}(x)}(x))d \leq 0, (JH(x))d = 0 \}$



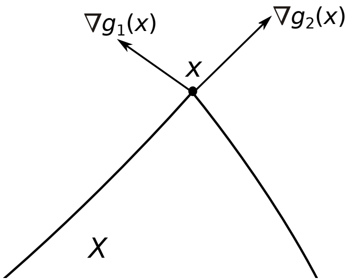
- $\mathcal{A}(x) \equiv$ zoom very close to x
- Each $i \in \mathcal{A}(x)$ defines "a part of ∂X"
- $\nabla g_i(x) \perp \partial X$ at x

- $T_X(x) = T_{X \cap \mathcal{B}(x, \varepsilon)}(x)$: only what happens as $x_i \to x$ matters
  $\implies$ if $x \notin \partial S(g_i, 0)$ then constraint $g_i(\cdot) \leq 0$ has no impact on $T_X(x)$
- $g_i \in C^0$: $x \in \partial S(g_i, 0) \implies g_i(x) = 0$, although $\not\Leftarrow$ (**check**)
- Active constraints at $x \in X$: $\mathcal{A}(x) = \{ i \in \mathcal{I} : g_i(x) = 0 \} \subseteq \mathcal{I}$
  "easy proxy" of $x \in \partial S(g_i, 0) \equiv$ constraint $g_i(\cdot) \leq 0$ impacts $T_X(x)$
- First-order feasible direction cone at $x \in X$: $D_X(x) =$
  $\{ d \in \mathbb{R}^n : \langle \nabla g_i(x), d \rangle \leq 0 \;\; i \in \mathcal{A}(x), \langle \nabla h_j(x), d \rangle = 0 \;\; j \in \mathcal{J} \} =$
  $\{ d \in \mathbb{R}^n : (JG_{\mathcal{A}(x)}(x))d \leq 0, (JH(x))d = 0 \}$



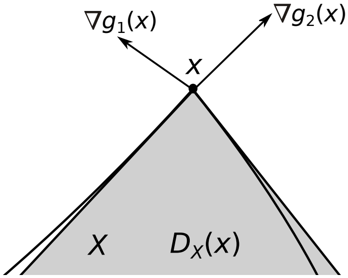
- $\mathcal{A}(x) \equiv$ zoom very close to x
- Each $i \in \mathcal{A}(x)$ defines "a part of ∂X"
- $\nabla g_i(x) \perp \partial X$ at x
- Each one separately $\implies$

- $T_X(x) = T_{X \cap \mathcal{B}(x\,,\,\varepsilon)}(x)$: only what happens as $x_i \to x$ matters
  $\Longrightarrow$ if $x \notin \partial S(g_i\,,\,0)$ then constraint $g_i(\cdot) \leq 0$ has no impact on $T_X(x)$
- $g_i \in C^0$: $x \in \partial S(g_i\,,\,0) \Longrightarrow g_i(x) = 0$, although $\not\Longleftarrow$ (**check**)
- Active constraints at $x \in X$: $\mathcal{A}(x) = \{\, i \in \mathcal{I} \,:\, g_i(x) = 0 \,\} \subseteq \mathcal{I}$
  "easy proxy" of $x \in \partial S(g_i\,,\,0) \equiv$ constraint $g_i(\cdot) \leq 0$ impacts $T_X(x)$
- First-order feasible direction cone at $x \in X$: $D_X(x) =$
  $\{\, d \in \mathbb{R}^n \,:\, \langle \nabla g_i(x)\,,\, d \rangle \leq 0 \;\; i \in \mathcal{A}(x)\,,\, \langle \nabla h_j(x)\,,\, d \rangle = 0 \;\; j \in \mathcal{J} \,\} =$
  $\{\, d \in \mathbb{R}^n \,:\, (\, JG_{\mathcal{A}(x)}(x)\,)d \leq 0\,,\, (\, JH(x)\,)d = 0 \,\}$



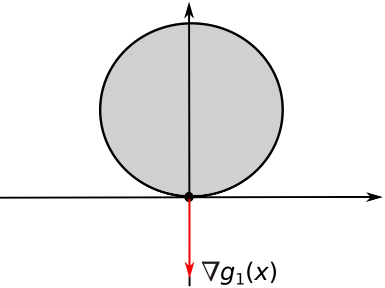
- $\mathcal{A}(x) \equiv$ zoom very close to x
- Each $i \in \mathcal{A}(x)$ defines "a part of ∂X"
- $\nabla g_i(x) \perp \partial X$ at x
- Each one separately $\Longrightarrow$ intersection

- $T_X(x) = T_{X \cap \mathcal{B}(x, \varepsilon)}(x)$: only what happens as $x_i \to x$ matters
  $\Longrightarrow$ if $x \notin \partial S(g_i, 0)$ then constraint $g_i(\cdot) \leq 0$ has no impact on $T_X(x)$
- $g_i \in C^0$: $x \in \partial S(g_i, 0) \Longrightarrow g_i(x) = 0$, although $\not\Longleftarrow$ (**check**)
- Active constraints at $x \in X$: $\mathcal{A}(x) = \{ i \in \mathcal{I} : g_i(x) = 0 \} \subseteq \mathcal{I}$
  "easy proxy" of $x \in \partial S(g_i, 0) \equiv$ constraint $g_i(\cdot) \leq 0$ impacts $T_X(x)$
- First-order feasible direction cone at $x \in X$: $D_X(x) =$
  $\{ d \in \mathbb{R}^n : \langle \nabla g_i(x), d \rangle \leq 0 \;\; i \in \mathcal{A}(x), \langle \nabla h_j(x), d \rangle = 0 \;\; j \in \mathcal{J} \} =$
  $\{ d \in \mathbb{R}^n : (JG_{\mathcal{A}(x)}(x))d \leq 0, (JH(x))d = 0 \}$



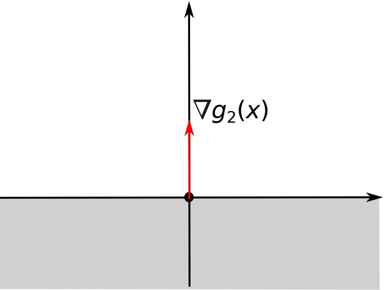
- $\mathcal{A}(x) \equiv$ zoom very close to x
- Each $i \in \mathcal{A}(x)$ defines "a part of ∂X"
- $\nabla g_i(x) \perp \partial X$ at x
- Each one separately $\Longrightarrow$ intersection
- $D_X(x) \supseteq T_X(x)$ but can be $\neq$

- $g_i \in C^1 \implies D_X(x) \supseteq T_X(x)$ [8, Lemma 12.2.(i)] (proof easy, just Taylor)
- $D_X(x)$ can be strictly larger than $T_X(x)$ in pathological cases



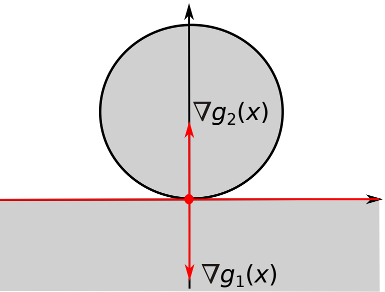
- $\min\{\ldots : x_1^2 + (x_2 - 1)^2 - 1 \leq 0$,

- $g_i \in C^1 \implies D_X(x) \supseteq T_X(x)$ [8, Lemma 12.2.(i)] (proof easy, just Taylor)
- $D_X(x)$ can be strictly larger than $T_X(x)$ in pathological cases



- $\min\{\ldots : x_1^2 + (x_2 - 1)^2 - 1 \leq 0 \ , \ x_2 \leq 0\}$

- $g_i \in C^1 \implies D_X(x) \supseteq T_X(x)$ [8, Lemma 12.2.(i)] (proof easy, just Taylor)
- $D_X(x)$ can be strictly larger than $T_X(x)$ in pathological cases



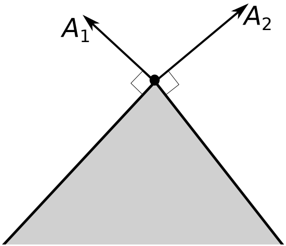
- $\min\{ \dots : x_1^2 + (x_2 - 1)^2 - 1 \leq 0 \; , \; x_2 \leq 0 \}$
  $\implies X = \{ x = [0, 0] \}$
- $D_X(x) = \{ [x_1, x_2] : x_2 = 0 \}$
- $T_X(x) = \{ [0, 0] \}$

Exercise: Check the counter-example in details

- A very stupid way to write $X = \{ [0, 0] \}$, have to avoid it
- Note that everything is convex, so convexity won't help this time

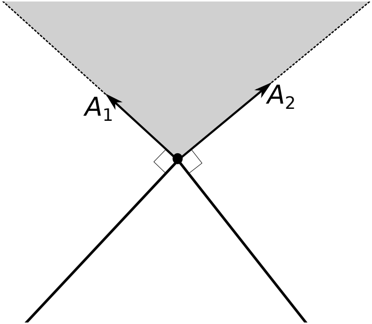
- Several conditions known $\equiv$ constraint qualifications:
  a) Affine constraints (AffC): g_i and h_j affine $\forall\, i \in \mathcal{I}$ and $j \in \mathcal{J} \implies$
     $T_X(\,x\,) = D_X(\,x\,)\ \forall\, x \in X$
  b) Slater's condition (SlaC): g_i convex $\forall\, i \in \mathcal{I}$, h_j affine $\forall\, j \in \mathcal{J}$
     $\exists\, \bar{x} \in X$ s.t. $g_i(\,\bar{x}\,) < 0\ \forall\, i \in \mathcal{I} \implies \quad T_X(\,x\,) = D_X(\,x\,)\ \forall\, x \in X$
  c) Linear independence (LinI): $\bar{x} \in X \wedge$ the vectors
     $\{\, \nabla g_i(\,\bar{x}\,) \,:\, i \in \mathcal{A}(\,\bar{x}\,) \,\} \;\cup\; \{\, \nabla h_j(\,\bar{x}\,) \,:\, j \in \mathcal{J} \,\}$
     all linearly independent from each other $\implies T_X(\,\bar{x}\,) = D_X(\,\bar{x}\,)$
- Weaker form of (SlaC): $\exists\, \bar{x} \in X$ s.t. $g_i(\,\bar{x}\,) < 0\ \forall i \in \mathcal{I}$ not affine $\equiv$ "in the interior of the feasible region of the nonlinear inequalities"
- Our counter-example fail all three (obviously)
- Wrap up: (AffC) $\vee$ ([w]SlaC) $\vee$ (LinI) $\implies$
  x local optimum $\implies \langle\, \nabla f(\,x\,)\,,\, d\,\rangle \geq 0 \quad \forall\, d \in D_X(\,x\,)$
- How do I check something like this? $\forall\, d$ ...??

- D_X is a polyhedral cone: $\mathcal{C} = \{ d \in \mathbb{R}^n : Ad \leq 0 \}$ for some $A \in \mathbb{R}^{k \times n}$
  [what about $\mathcal{J}$? (**check**)] "very close by, ∂X looks like a polyhedron"



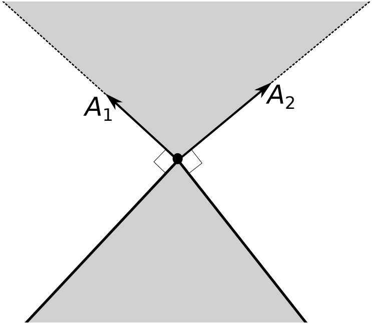
Farkas' lemma: a tale of two cones [8, p. 326]

- D_X is a polyhedral cone: $\mathcal{C} = \{ d \in \mathbb{R}^n : Ad \leq 0 \}$ for some $A \in \mathbb{R}^{k \times n}$
  [what about $\mathcal{J}$? (**check**)] "very close by, ∂X looks like a polyhedron"



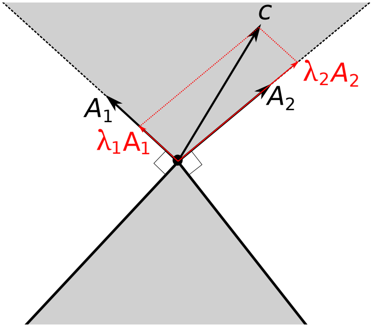
- Dual cone: $\mathcal{C}^* = \{ c = \sum_{i=1}^{k} \lambda_i A_i : \lambda \geq 0 \}$

- D_X is a polyhedral cone: $\mathcal{C} = \{\, d \in \mathbb{R}^n \,:\, Ad \le 0 \,\}$ for some $A \in \mathbb{R}^{k\times n}$
  [what about $\mathcal{J}$? (**check**)] "very close by, ∂X looks like a polyhedron"



- Dual cone: $\mathcal{C}^* = \{\, c = \sum_{i=1}^{k} \lambda_i A_i \,:\, \lambda \ge 0 \,\}$
- Farkas' lemma:

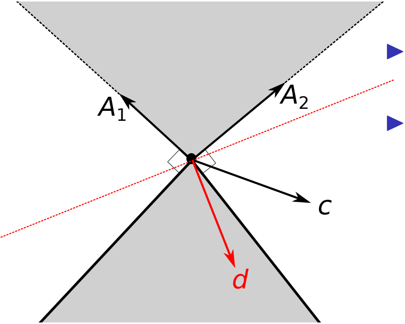
- D_X is a polyhedral cone: $\mathcal{C} = \{ d \in \mathbb{R}^n \, : \, Ad \leq 0 \}$ for some $A \in \mathbb{R}^{k \times n}$
  [what about $\mathcal{J}$? (**check**)] "very close by, ∂X looks like a polyhedron"



- Dual cone: $\mathcal{C}^* = \{ c = \sum_{i=1}^{k} \lambda_i A_i \, : \, \lambda \geq 0 \}$
- Farkas' lemma: $\forall \, c \in \mathbb{R}^n$
  either $\exists \, \lambda \geq 0$ s.t. $c = \lambda^T A = \sum_{i=1}^{k} \lambda_i A_i$

Farkas' lemma: a tale of two cones [8, p. 326]

- D_X is a polyhedral cone: $\mathcal{C} = \{\, d \in \mathbb{R}^n \ : \ Ad \le 0 \,\}$ for some $A \in \mathbb{R}^{k \times n}$
  [what about $\mathcal{J}$? (**check**)] "very close by, ∂X looks like a polyhedron"

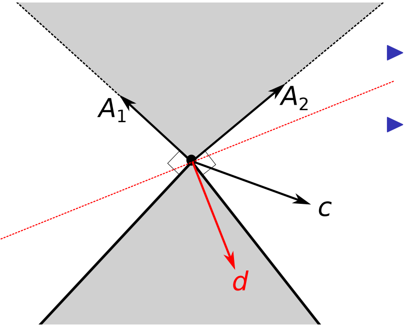


- Dual cone: $\mathcal{C}^* = \{\, c = \sum_{i=1}^k \lambda_i A_i \ : \ \lambda \ge 0 \,\}$
- Farkas' lemma: $\forall\, c \in \mathbb{R}^n$

  either $\quad \exists\, \lambda \ge 0$ s.t. $c = \lambda^T A = \sum_{i=1}^k \lambda_i A_i$

  or $\quad \exists\, d$ s.t. $Ad \le 0 \ \wedge \ \langle\, c \,,\, d \,\rangle > 0$

- D_X is a polyhedral cone: $\mathcal{C} = \{ d \in \mathbb{R}^n \,:\, Ad \leq 0 \}$ for some $A \in \mathbb{R}^{k \times n}$
  [what about $\mathcal{J}$? (**check**)] "very close by, ∂X looks like a polyhedron"



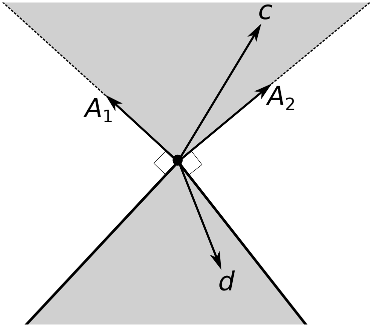
- Dual cone: $\mathcal{C}^* = \{ c = \sum_{i=1}^{k} \lambda_i A_i \,:\, \lambda \geq 0 \}$
- Farkas' lemma: $\forall\, c \in \mathbb{R}^n$

  either $\quad \exists\, \lambda \geq 0$ s.t. $c = \lambda^T A = \sum_{i=1}^{k} \lambda_i A_i$

  or $\quad \exists\, d$ s.t. $Ad \leq 0 \,\wedge\, \langle\, c\,,\, d \,\rangle > 0$

  (one and only one of these is true)

- D_X is a polyhedral cone: $\mathcal{C} = \{ d \in \mathbb{R}^n : Ad \leq 0 \}$ for some $A \in \mathbb{R}^{k \times n}$
  [what about $\mathcal{J}$? (**check**)] "very close by, ∂X looks like a polyhedron"



- Dual cone: $\mathcal{C}^* = \{ c = \sum_{i=1}^{k} \lambda_i A_i : \lambda \geq 0 \}$
- Farkas' lemma: $\forall\, c \in \mathbb{R}^n$
  either $\exists\, \lambda \geq 0$ s.t. $c = \lambda^T A = \sum_{i=1}^{k} \lambda_i A_i$
  or $\exists\, d$ s.t. $Ad \leq 0 \,\wedge\, \langle c , d \rangle > 0$
  (one and only one of these is true)

- $\langle c , d \rangle \leq 0 \;\forall\, d \in \mathcal{C}$, $c \in \mathcal{C}^*$ (**check**) actually a $\neq$ definition:
  polar cone $\mathcal{C}^\circ = \{ c \in \mathbb{R}^n : \langle c , d \rangle \leq 0 \;\forall\, d \in \mathcal{C} \} = \mathcal{C}^* \implies$
  Farkas' lemma $\equiv$ either $c \in \mathcal{C}^*$, or $c \notin \mathcal{C}^*$

Exercise: Farkas' lemma the father of all separation results: $x \notin X$ (convex) $\implies \exists$ an hyperplane that separates x from X. Discuss.

Farkas' lemma $\rightarrow$ KKT [8, §12.3][7, p. 342]

- (CQ $\wedge$) x_* optimum $\Longrightarrow \langle \nabla f(x_*), d \rangle \geq 0 \; \forall d$ s.t.

$$\langle \nabla g_i(x_*), d \rangle \leq 0 \;\; i \in \mathcal{A}(x_*) \quad , \quad \langle \nabla h_j(x_*), d \rangle = 0 \;\; j \in \mathcal{J}$$

- (CQ $\wedge$) x_* optimum $\implies \langle \nabla f(x_*) , d \rangle \geq 0 \; \forall\, d$ s.t.

  $\langle \nabla g_i(x_*) , d \rangle \leq 0 \;\; i \in \mathcal{A}(x_*) \quad , \quad \langle \nabla h_j(x_*) , d \rangle = 0 \;\; j \in \mathcal{J}$

  $\equiv \exists\, \lambda \in \mathbb{R}_+^{\#\mathcal{A}(x_*)}$ and $\mu \in \mathbb{R}^{\#\mathcal{J}}$ s.t.

  $$\nabla f(x_*) + \sum_{i \in \mathcal{A}(x_*)} \lambda_i \nabla g_i(x_*) + \sum_{j \in \mathcal{J}} \mu_j \nabla h_j(x_*) = 0 \qquad \text{(GC)}$$

Exercise: check details: why the sign of $\nabla f(x_*)$? Why $\mu \ngeq 0$?

- Constructive way to prove necessary condition: find λ and μ

- Karush-Kuhn-Tucker conditions: $\exists\, \lambda \in \mathbb{R}_+^{\#\mathcal{I}}$ and $\mu \in \mathbb{R}^{\#\mathcal{J}}$ s.t.

  $$g_i(x) \leq 0 \;\; i \in \mathcal{I} \quad , \quad h_j(x) = 0 \;\; j \in \mathcal{J} \qquad \text{(KKT-F)}$$

  $$\nabla f(x) + \sum_{i \in \mathcal{I}} \lambda_i \nabla g_i(x) + \sum_{j \in \mathcal{J}} \mu_j \nabla h_j(x) = 0 \qquad \text{(KKT-G)}$$

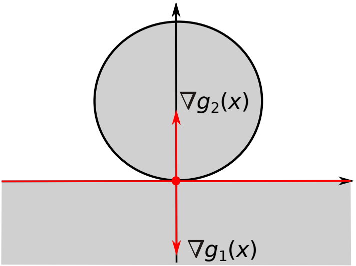
  $$\sum_{i \in \mathcal{I}} \lambda_i g_i(x) = 0 \qquad \text{(KKT-CS)}$$

- (KKT-CS) = Complementary Slackness $\equiv \lambda_i g_i(x) = 0 \; \forall\, i \in \mathcal{I}$

Exercise: prove the statement above and explain where (KKT-CS) comes from

- KKT Theorem: $T_X(x) = D_X(x) \wedge x$ local optimum $\implies$ (KKT)
- Optimization $\equiv$ solving systems of nonlinear equations and inequalities

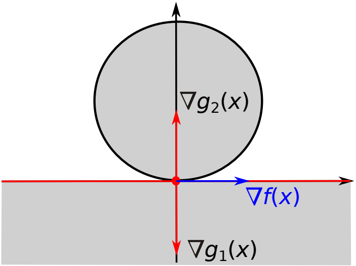
- KKT Theorem: $T_X(x) = D_X(x) \wedge x$ local optimum $\implies$ (KKT)
- Optimization $\equiv$ solving systems of nonlinear equations and inequalities



- $T_X(x) = D_X(x)$ crucial: counter-example

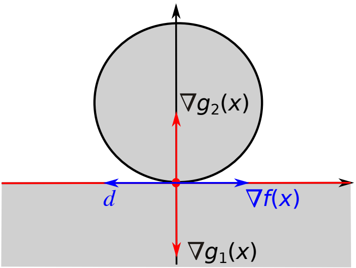
  $\min\{\, x_1 \;:\; x_1^2 + (x_2 - 1)^2 - 1 \leq 0 \;\;,\;\; x_2 \leq 0 \,\}$

- KKT Theorem: $T_X(x) = D_X(x) \wedge x$ local optimum $\Longrightarrow$ (KKT)
- Optimization $\equiv$ solving systems of nonlinear equations and inequalities



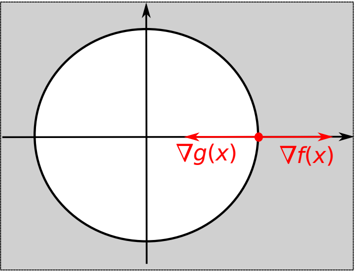
- $T_X(x) = D_X(x)$ crucial: counter-example
  $\min\{ x_1 \,:\, x_1^2 + (x_2 - 1)^2 - 1 \leq 0 \;,\; x_2 \leq 0 \}$

- KKT Theorem: $T_X(x) = D_X(x) \wedge x$ local optimum $\implies$ (KKT)
- Optimization $\equiv$ solving systems of nonlinear equations and inequalities



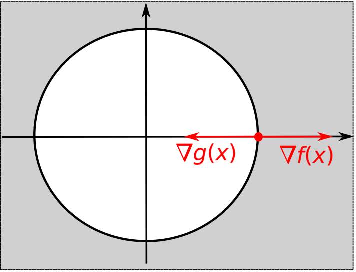
- $T_X(x) = D_X(x)$ crucial: counter-example
  $\min\{ x_1 \,:\, x_1^2 + (x_2 - 1)^2 - 1 \leq 0 \;,\; x_2 \leq 0 \}$
  (x optimum but (KKT) do not hold)

- KKT Theorem: $T_X(x) = D_X(x) \wedge x$ local optimum $\Longrightarrow$ (KKT)
- Optimization $\equiv$ solving systems of nonlinear equations and inequalities



- $T_X(x) = D_X(x)$ crucial: counter-example
  $\min\{\, x_1 \,:\, x_1^2 + (x_2 - 1)^2 - 1 \leq 0 \;,\; x_2 \leq 0 \,\}$
  (x optimum but (KKT) do not hold)
- Condition not necessary: counter-example
  $\min\{\, x_1 \,:\, x_1^2 + x_2^2 \geq 1 \,\} \;,\; x = [\,1\,,\,0\,]$
  (∇f cannot tell maxima from minima)

- KKT Theorem: $T_X(x) = D_X(x) \wedge x$ local optimum $\Longrightarrow$ (KKT)
- Optimization $\equiv$ solving systems of nonlinear equations and inequalities



- $T_X(x) = D_X(x)$ crucial: counter-example
  $\min\{\, x_1 \,:\, x_1^2 + (x_2 - 1)^2 - 1 \leq 0 \;\;,\;\; x_2 \leq 0 \,\}$
  (x optimum but (KKT) do not hold)
- Condition not necessary: counter-example
  $\min\{\, x_1 \,:\, x_1^2 + x_2^2 \geq 1 \,\} \;\;,\;\; x = [\,1\,,\,0\,]$
  (∇f cannot tell maxima from minima)

- Only safe case: no maxima $\equiv$ (P) convex problem: f convex on X,
  X convex $\Longleftarrow$ $g_i(x)$ convex $\forall i \in \mathcal{I}$ $\;\;,\;\;$ $h_j(x)$ affine $\forall j \in \mathcal{J}$

- For (P) convex, (KKT) $\Longrightarrow$ x global optimum:
  (KKT) $\Longrightarrow \langle \nabla f(x), d \rangle \geq 0 \;\forall\, d \in D_X(x)$ and $D_X(x) \supseteq T_X(x) \supseteq F_X(x)$
  $\Longrightarrow \langle \nabla f(x), d \rangle \geq 0 \;\forall\, d \in F_X(x) \Longrightarrow x$ global optimum

Outline

- (P) not convex $\equiv$ (KKT) not sufficient $\implies$ have to use second-order
- But clearly cannot be just "$\nabla^2 f(x_*) \succeq 0$"
- Fundamental concept: Lagrangian function

  $L(x\,;\lambda\,,\mu) = f(x) + \sum_{i \in \mathcal{I}} \lambda_i g_i(x) + \sum_{j \in \mathcal{J}} \mu_j h_j(x)$

  x variables, λ and μ parameters
- Fundamental observation: $(x\,,\lambda\,,\mu)$ satisfies (KKT-G) $\equiv$

  $\nabla L(x\,;\lambda\,,\mu) = 0$ (gradient on x alone)

  $\implies x$ stationary point of $L(\cdot\,;\lambda\,,\mu)$
- When is stationary point also a minimum?
- One might guess "$\nabla^2 L(x\,;\lambda\,,\mu) \succeq 0$" to be the answer: almost, but not quite

- Assume (x, λ, μ) satisfies (KKT): critical cone $\subseteq \mathbb{R}^n$

$$C(x; \lambda, \mu) = \left\{ d \in \mathbb{R}^n : \begin{array}{ll} \langle \nabla g_i(x), d \rangle = 0 & i \in \mathcal{A}(x) \text{ s.t. } \lambda_i^* > 0 \\ \langle \nabla g_i(x), d \rangle \leq 0 & i \in \mathcal{A}(x) \text{ s.t. } \lambda_i^* = 0 \\ \langle \nabla h_j(x), d \rangle = 0 & i \in \mathcal{J} \end{array} \right\}$$

- (x, λ, μ) satisfies (KKT) $\wedge\, x$ satisfies (LinI): x local optimum $\Longrightarrow$

  $$d^T \nabla^2 L(x; \lambda, \mu) d \geq 0 \qquad \forall\, d \in C(x; \lambda, \mu)$$

  "the Hessian of the Lagrangian function is $\succeq 0$ on the critical cone"

- $(x; \lambda, \mu)$ satisfies (KKT) $\wedge\; \nabla^2 L(x; \lambda, \mu) \succ 0$ on $C(x, \lambda, \mu)$

  $\Longrightarrow$ x local optimum (sufficient)

- Conditions for unconstrained optimization a special case (**check**)

- Hardly anybody cares

Outline

- Lagrangian function interesting object: objective and constraints together
- (KKT-G) $\equiv$ x stationary point of $L(\cdot)$ for the right $\lambda \geq 0$ and μ
- Assume we know the right $\lambda \geq 0$ and μ: can we find x?

- Lagrangian function interesting object: objective and constraints together
- (KKT-G) $\equiv$ x stationary point of $L(\cdot)$ for the right $\lambda \geq 0$ and μ
- Assume we know the right $\lambda \geq 0$ and μ: can we find x?
- Natural idea: solve the Lagrangian relaxation

  $(R_{\lambda,\mu}) \quad \psi(\lambda\,,\,\mu) = \min_x \{\, L(x\,;\,\lambda\,,\,\mu) \,:\, x \in \mathbb{R}^n \,\}$

- Lagrangian function interesting object: objective and constraints together
- (KKT-G) $\equiv$ x stationary point of $L(\cdot)$ for the right $\lambda \geq 0$ and μ
- Assume we know the right $\lambda \geq 0$ and μ: can we find x?
- Natural idea: solve the Lagrangian relaxation

$$(R_{\lambda,\mu}) \quad \psi(\lambda, \mu) = \min_x \{ L(x; \lambda, \mu) : x \in \mathbb{R}^n \}$$

- Er ... "relaxation"? This a Yoga course now perchance?

- Lagrangian function interesting object: objective and constraints together
- (KKT-G) $\equiv$ x stationary point of $L(\cdot)$ for the right $\lambda \geq 0$ and μ
- Assume we know the right $\lambda \geq 0$ and μ: can we find x?
- Natural idea: solve the Lagrangian relaxation

  $$(R_{\lambda,\mu}) \quad \psi(\lambda, \mu) = \min_x \{ L(x; \lambda, \mu) : x \in \mathbb{R}^n \}$$

- (P) $\min\{ f(x) : x \in X \}$, relaxation = another optimization problem carefully constructed in order to provide a lower bound on $\nu(P)$

- Lagrangian function interesting object: objective and constraints together
- (KKT-G) $\equiv$ x stationary point of $L(\cdot)$ for the right $\lambda \geq 0$ and μ
- Assume we know the right $\lambda \geq 0$ and μ: can we find x?
- Natural idea: solve the Lagrangian relaxation

  $(R_{\lambda,\mu}) \quad \psi(\lambda, \mu) = \min_x \{ L(x; \lambda, \mu) : x \in \mathbb{R}^n \}$
- (P) $\min\{ f(x) : x \in X \}$, relaxation = another optimization problem carefully constructed in order to provide a lower bound on $\nu(P)$
- $(\underline{P})$ $\min\{ \underline{f}(x) : x \in \underline{X} \}$ a relaxation of (P) $\Longrightarrow$ $\nu(\underline{P}) \leq \nu(P)$ if

  i) $\underline{X} \supseteq X$, ii) $\underline{f}(x) \leq f(x) \; \forall x \in X$
- $(R_{\lambda,\mu})$ is a relaxation of (P) $\forall \lambda \geq 0$ and μ (**check**) $\Longrightarrow$

  weak duality: $\forall x \in X$, $\psi(\lambda, \mu) \leq \nu(P) \leq f(x)$
- But how do I choose $\lambda \geq 0$ and μ?

- Dual function ψ is nice-ish:
  1. often easy to compute: $(R_{\lambda,\mu})$ unconstrained problem
  2. ψ concave (**check**), but note that $\psi(\lambda, \mu) = -\infty$ happens
  3. $\bar{x}$ optimal in $(R_{\lambda,\mu}) \implies [\, G(\bar{x}) , H(\bar{x}) \,] \in \partial\psi(\lambda, \mu)$ [6, Prop. XII.2.2.2]
  4. $\psi \notin C^1$ even if f, g_i, $h_j \in C^1$, but $\bar{x}$ unique optimal solution to $(R_{\lambda,\mu})$ $\implies \psi$ is differentiable in (λ, μ), $\nabla\psi(\lambda, \mu) = [\, G(\bar{x}) , H(\bar{x}) \,]$ [6, p. 156]
- 1. – 3. $\implies \psi$ "easy" to maximize $\implies$ Lagrangian dual of (P):

  (D) $\max\{\, \psi(\lambda, \mu) : \lambda \in \mathbb{R}_+^{|\mathcal{I}|} \,,\; \mu \in \mathbb{R}^{|\mathcal{J}|} \,\}$

  a convex program (not unconstrained, but constraints very easy: $\lambda \geq 0$) that gives a lower bound $\nu(D) \leq \nu(P)$ even if (P) not convex
- No free lunch: $\psi(\cdot) = \nu(R_{\lambda,\mu})$ need be solved to global optimality and not (necessarily) convex, but if you can do that everything works even if (P) "ferociously conconvex" (e.g., $x \in \mathbb{Z}^n$ constraints)
- How good is the bound $\nu(D)$? When is $\nu(D) = \nu(P)$ ("strong duality")?

- Not always, but yes if (P) convex (and regular)

- (P) convex, x_* optimum, $T_X(x_*) = D_X(x_*) \implies \nu(D) = \nu(P)$
  under regularity, convex programs always have strong duality

- Under further conditions, solving (D) actually solves (P) [8, Th. 12.13]:
  (R_{λ_*, μ_*}) has unique minumum $x_* \implies x_*$ optimum of (P)

Exercise: Suggest conditions on (P) so that (R_{λ_*, μ_*}) has unique miniumum

- (R_{λ_*, μ_*}) has multiple minima $\implies$ not all optimal (not even feasible)
  but recovering x_* from (λ_*, μ_*) most often doable (will see soon)

- Duality a powerful alternative for solving constrained convex problems

- Duality fundamental to compute valid lower bounds for nonconvex problems

- Counter-example: $\min\{\, -x^2 \,:\, 0 \leq x \leq 1 \,\}$
  $L(\, x \,,\, \lambda \,) = -x^2 + \lambda^1(x-1) - \lambda^2 x \quad , \quad \psi(\, \lambda \,) = \min_{x \in \mathbb{R}} L(\, x \,,\, \lambda \,)$
  $\psi(\, \lambda \,) = -\infty \;\forall\, \lambda \in \mathbb{R}^2_+ \quad \Longrightarrow \; \nu(\, D \,) = -\infty < \nu(\, P \,) = -1$

- Note: $x_* = 1$, $\lambda^1_* = 2$, $\lambda^2_* = 0 \;\Longrightarrow\; -2x_* + \lambda^1_* - \lambda^2_* = 0 \;\equiv\;$ KKT,
  but x^* maximum of (R_{λ_*,μ_*}) (stationary, not minumum)

- Counter-example is nonconvex, convexity (and regularity) does help here

- (P) convex, x_* optimum, $T_X(\, x_* \,) = D_X(\, x_* \,) \;\Longrightarrow\; \nu(\, D \,) = \nu(\, P \,)$
  Proof: Since x_* optimum, necessary conditions hold
  but $T_X(\, x_* \,) = D_X(\, x_* \,) \;\Longrightarrow\; \exists\, [\, \lambda_* \,,\, \mu_* \,]$ satisfying KKT with x_*
  Claim: $[\, \lambda_* \,,\, \mu_* \,]$ optimal solution to (D), and $\nu(\, D \,) = \nu(\, P \,)$
  x_* stationary for (R_{λ_*,μ_*}) + everything convex $\;\Longrightarrow\;$ x_* optimal $\;\Longrightarrow$
  $\nu(\, D \,) \geq \psi(\, \lambda_* \,,\, \mu_* \,) = L(\, x_* \,;\, \lambda_* \,,\, \mu_* \,) = f(\, x_* \,) = \nu(\, D \,) \geq \nu(\, P \,)$

Exercise: prove x_* stationary and $L(\, x_* \,;\, \lambda^* \,,\, \mu^* \,) = f(\, x^* \,)$

- (P) $\min\{\, f(\,x\,) \,:\, G(\,x\,) \leq 0 \,\}$ convex and regular, $\exists$ dual optimal solution λ_*
- $(P) = (P_0)$, where (P_r) $\phi(\,r\,) = \min\{\, f(\,x\,) \,:\, G(\,x\,) \leq r \,\}$ $: \mathbb{R}^m \to \mathbb{R}$
- Every $g_i(\,x\,) \leq 0$ is a resource limiting my output = the money I can save: how much money would it save me to have ε more of resource i?
- $\phi(\cdot)$ convex [6, p. 179], $-\lambda_* \in \partial\phi(\,0\,) \;\equiv\; \phi(\,v\,) - \phi(\,0\,) \geq -\langle\, v \,,\, \lambda_* \,\rangle$ (**check**)
- Consider $v = u^i$ = buying one more unit of resource i and nothing else
- (KKT-CS) $\equiv$ resource i not fully used $\Longrightarrow$ $\lambda^i_* = 0 \;\equiv\; \phi(\,u^i\,) \geq \phi(\,0\,)$: any more of resource i cannot decrease $v(P)$, has no value to me
- $\lambda^i_* > 0 \;\Longrightarrow\; \phi(\,u^i\,) \geq \phi(\,0\,) - \lambda^i_*$: $v(P)$ decreases at most λ^i_* (may be less) $\Longrightarrow$ maximum price I should buy at ("shadow price") = value of resource i
- Useful for sensitivity analysis: what happens if my data is (a bit) wrong
- Useful to economists (who would love the world being convex, but it's not)
- Very useful for algorithms, will see

Outline

- Cumbersome max / min in Lagrangian dual simplifies to max in special cases

- Linear Program $(P)\ \ \min\{\, cx \,:\, Ax \geq b \,\} \implies$

  Lagrangian function $L(\, x\,;\, \lambda\,) = cx + \lambda(\, b - Ax\,) = \lambda b + (c - \lambda A)x \implies$

  Lagrangian relaxation $(R_\lambda)\ \ \lambda b + \min\{\, (c - \lambda A)x \,:\, x \in \mathbb{R}^n \,\}$ so simple it can be solved by closed formula $\psi(\,\lambda\,) = \nu(\, R_\lambda\,) = \begin{cases} -\infty & \text{if } c - \lambda A \neq 0 \\ \lambda b & \text{if } c - \lambda A = 0 \end{cases}$

- $(D)\ \ \max\,\{\, \psi(\,\lambda\,) \,:\, \lambda \geq 0 \,\} \ = \ \max\,\{\, \lambda b \,:\, \lambda A = c \,,\ \lambda \geq 0 \,\}$
  a $\neq$ Linear Program (variables $\longleftrightarrow$ constraints) with the same data

Exercise: what is the dual of (D)?

Exercise: what if $(P)\ \ \min\{\, cx \,:\, Ax \geq b \,,\, x \geq 0 \,\}$? what if it has any form?

- Strong duality $\equiv$ $\nu(\, D\,) = \nu(\, P\,)$ (almost) always holds

Exercise: prove last statement. Why "almost"? Can $\nu(\, D\,) > \nu(\, P\,)$ happen?

1. Very simple Quadratic Program (QP): $(P)\;\; \min\{\, \frac{1}{2}\|\, x \,\|_2^2 \;:\; Ax = b \,\}$
   - Lagrangian function: $L(\, x \,;\, \mu \,) = \frac{1}{2}\|\, x \,\|_2^2 + \mu(\, Ax - b \,) =$
     $= -\mu b + [\, R_\mu(\, x \,) = \frac{1}{2}\|\, x \,\|_2^2 + (\, \mu A \,)x \,]$ (singling out what depends on x)
   - Dual function $\psi(\, \mu \,) = \min_{x \in \mathbb{R}^n} L(\, x \,;\, \mu \,) = -\mu b + \min_{x \in \mathbb{R}^n}\{\, R_\mu(\, x \,) \,\}$
     $\nabla R_\mu(\, x \,) = x + \mu A = 0 \iff x = -\mu A \implies \psi(\, \mu \,) = -\frac{1}{2}\mu^T(AA^T)\mu - \mu b$
   - $(D)\;\; \max\{\, -\frac{1}{2}\mu^T(AA^T)\mu - \mu b \;:\; \mu \in \mathbb{R}^m \,\}$ (an unconstrained QP)

2. Strictly convex QP: $Q \succ 0$, $(P)\;\; \min\{\, \frac{1}{2}x^T Qx + qx \;:\; Ax \geq b \,\}$
   - Strong duality $\equiv \nu(\, D \,) = \nu(\, P \,)$ (almost) always holds with
     $(D)\;\; \max\{\, \lambda b - \frac{1}{2}v^T Q^{-1} v \;:\; \lambda A - v = q \,,\, \lambda \geq 0 \,\}$

Exercise: prove last $(P) \rightsquigarrow (D)$. What would change if (P) had $Ax = b$?

Exercise: compute (D) when "some variables are not quadratic": $x = [\, z \,,\, w \,]$,
objective $\frac{1}{2}z^T Qz + qz + pw$ with $Q \succ 0$, constraints $Az + Ew \geq b$

Exercise: compute (D) when $Q \succeq 0$ but is singular: what would happen if $Q \not\succeq 0$?

Conic Programs [2, §4.6][11][12]

- Conic Program: $(P) \;\; \min\{\, cx \,:\, Ax \geq_K b \,\}$

  where $x \geq_K y \;\equiv\; x - y \in K$ with K pointed convex cone, e.g.
  - $K = \mathbb{R}^n_+ \;\equiv\;$ sign constraints $\;\equiv\;$ Linear Program
  - $K = \mathbb{L} = \{\, x \in \mathbb{R}^n \,:\, x_n \geq \sqrt{\sum_{i=1}^{n-1} x_i^2} \,\} \;\equiv\;$ Second-Order Cone Program
  - $K = \mathbb{S}_+ = \{\, Q \in \mathbb{R}^{n \times n} \,:\, Q \succeq 0 \,\} \;\equiv\;$ "$\succeq$" constraints $\;\equiv\;$ SemiDefinite Program

  or any combination of the three
- Exceedingly smart idea: everything is linear, but the cone is not

  $\equiv$ a nonlinear program disguised as a linear one
- Any LP and convex QP is a SOCP, vice-versa is not true

Exercise: prove $\|\, x \,\|_2^2 \,/\, s \leq t \;\;\equiv\;\; \big|\big|\, [\, x \,,\, (\, t - s \,) \,/\, 2\,] \,\big|\big| \leq (\, t + s \,) \,/\, 2$,
discuss why it proves the above statement

- Any SOCP is a SDP, vice-versa is not true

Exercise: prove any SOCP is a SDP, vice-versa not true easy to see, hard to prove

- Conic Dual: $(D)\ \ \max\{\,\lambda b\ :\ \lambda A = c\,,\ \lambda \geq_{K^D} 0\,\}$
  where $K^D = \{\, z\ :\ \langle\, z\,,\ v\,\rangle \geq 0\ \ \forall\, v \in K\,\}$ dual cone
  ($\neq$ definition from before, actually $K^D = -K^\circ$)

- Another Conic Program with the same data (but $\neq$ cone)

- Except all three cones above are self-dual: $K^D = K$
  "the angle at the vertex of the cone is 90 degrees"

Exercise: prove (P) $\rightsquigarrow$ (D) for general Conic Programs

- Strong duality not always holds, constraint qualification needed
  one of the constraints is nonlinear, even if it does not look so

- "Explicit form" of SOCP: $\min \{ cx : \| D_i x - d_i \| \leq p_i x - q_i \quad i = 1, \ldots, m \}$
  "explicit data" D_i, d_i, p_i, q_i (any LP is a SOCP: $D_i = 0$, $d_i = 0$)

- SOCP Dual written in terms of explicit data:
  $\max \{ \sum_{i=1}^m \lambda_i d_i + \nu_i q_i : \sum_{i=1}^m \lambda_i D_i + \nu_i p_i = c \,,\ \| \lambda_i \| \leq \nu_i \ \ i = 1, \ldots, m \}$

- "Explicit form" of SDP: $\min \{ cx : \sum_{i=1}^n x_i A^i \succeq B \}$
  A^i, $B \in \mathbb{R}^{k \times k}$, k possibly $\neq n$, symmetric but not necessarily $\succeq 0$

- SDP Dual written in terms of explicit data:
  $\max \{ \langle B \,, \Lambda \rangle : \langle A^i \,, \Lambda \rangle = c_i \ \ i = 1, \ldots, n \,,\ \Lambda \succeq 0 \,,\ \ \Lambda \in \mathbb{R}^{k \times k} \}$
  where $\langle A \,, B \rangle = \sum_i \sum_j A_{ij} B_{ij}$ (Frobenius scalar product)

- Close to (but not exactly) "take LP duality and replace $\geq$ with the other cone"

- In all cases, formal algebraic rules that can be automated [10, 12]

Outline

- Nondifferentiable optimization, Proximal Bundle Method recall:
  - $\{ x^i \} \implies$ bundle $\mathscr{B}^i = \{ (x^h , f^h = f(x^h) , g^h \in \partial f(x^h)) \}_{h<i}$
  - $f^i_{\mathscr{B}}(x) = \max\{ f^h + \langle g^h , x - x^h \rangle \ : \ (x^h , f^h , g^h) \in \mathscr{B}^i \}$, (CP) model $f^i_{\mathscr{B}} \leq f$
  - stabilized (translated) master problem $\min\{ f_{\mathscr{B}}(\bar{x} + d) - f(\bar{x}) + \mu \| d \|^2 / 2 \}$

- $f^h - f(\bar{x}) + \langle g^h , x - x^h \rangle = [f^h - f(\bar{x}) + \langle g^h , \bar{x} - x^h \rangle] + \langle g^h , x - \bar{x} \rangle =$ $\langle g^h , d \rangle - \alpha^h$, $\alpha^h = f(\bar{x}) - f^h - \langle g^h , \bar{x} - x^h \rangle \geq 0$ linearization error w.r.t. $\bar{x}$

- $(PM_{\mathscr{B},\bar{x},\mu})$ $\min \{ v + \mu \| d \|^2 / 2 \, ; \, v \geq \langle g^h , d \rangle - \alpha^h \ \ h \in \mathscr{B} \} \rightsquigarrow$
  $(DM_{\mathscr{B},\bar{x},\mu})$ $[-] \min \{ 1 / (2\mu) \| \sum_{h \in \mathscr{B}} g^h \theta_h \|^2 + \sum_{h \in \mathscr{B}} \alpha^h \theta_h \ : \ \theta \in \Theta \}$
  with $\Theta = \{ \theta \in \mathbb{R}^{\#\mathscr{B}}_{+} \ : \ \sum_{h \in \mathscr{B}} \theta_h = 1 \}$ unitary simplex (convex)

- θ^* optimal for $(DM_{\mathscr{B},\bar{x},\mu})$, $z^* = \sum_{h \in \mathscr{B}} g^h \theta^*_h$, $\sigma^* = \sum_{h \in \mathscr{B}} \alpha^h \theta^*_h$ $[\geq 0] \implies$
  $[v^* , d^*] = [-(1 / \mu) \| z^* \| - \sigma^* , -(1 / \mu) z^*]$ optimal for $(PM_{\mathscr{B},\bar{x},\mu})$

Exercise: prove the statements above

- Solving $(DM_{\mathscr{B},\bar{x},\mu})$ may be faster than solving $(PM_{\mathscr{B},\bar{x},\mu})$ [3]
- $\theta^*_h = 0 \implies (\, g^h \,,\, \alpha^h \,)$ can be eliminated from $\mathscr{B}$ without losing convergence
- $(\, z^* \,,\, \sigma^* \,)$ is added to $\mathscr{B} \implies$ everything else can be eliminated
- $\mathscr{B} = \{\, (\, z^* \,,\, \sigma^* \,)\,\}$ "poorman's bundle" $\implies$ Proximal Bundle $\approx$ subgradient $\equiv$ slow but with a working stopping criterion (in theory)
- Exact stopping condition (extends to approximate): $d^* = 0 \;\wedge\; v^* = 0$ $\equiv\; z^* = \theta^* G = 0 \;,\; \sigma^* = 0 \;\equiv\; \alpha^h = 0 \;\forall\, h$ s.t. $\sigma^*_h > 0$
- Lagrangian case with $H(\, x\,) = Ax - b$ only: $d^* = 0 \implies$ all x^h s.t. $\theta^*_h > 0$ are optimal for $(R_{\lambda^*}) \implies x_* = \sum_{h \in \mathscr{B}} x^h \theta^*_h$ is optimal for (P)
- Extends to nonlinear $G(\, x\,) \leq 0$ & to ε-optimal $(\|\, d^* \,\| \leq \delta \;,\; \sigma^* \leq \varepsilon)$

Exercise: prove the statements above

- Solving (D) completely equivalent to solving (P) in the convex case, otherwise solving a convexified relaxation (possibly "tight") [4]

- Usual (scalar) learning setup: $I = \{ 1, \ldots, m \}$ set of samples $\equiv$ $X = [x^i \in \mathbb{R}^h]_{i \in I}$ inputs, $y = [y^i \in \mathbb{R}^1]_{i \in I}$ outputs, "explain" y from X
- $y^i \in \{ 1 , -1 \} \equiv$ classification: Support Vector Machine (SVM)

  $$\min_{w,b} \{ \| w \|^2 + C[\mathcal{L}(w , b) = \sum_{i \in I} \max\{ 1 - y^i(\langle w , x^i \rangle - b) , 0 \}] \}$$

  hyperparameter C weighs empiric loss against margin = regularization
- $\mathcal{L} \implies$ objective convex but nondifferentiable
- Extends to y^i arbitrary $\equiv$ Support Vector Regression (SVR)

  $$\min_{w,b} \{ \| w \|^2 + C[\mathcal{L}_\varepsilon(w , b) = \sum_{i \in I} \max\{ | \langle w , x^i \rangle - b - y^i | - \varepsilon , 0 \}] \}$$

  further hyperparameter ε controlling the "insensitivity tube"
  (two hyperparameters is not megl che one, grid search cost = grid size^2)
- $\mathcal{L}_\varepsilon \implies$ objective still convex but nondifferentiable
- Very specific nondifferentiability: max of linear functions (recall smoothing)
- Linear constraints can be better than a nondifferentiable objective

- Reformulation of SVM / SVR as a QP via "slack variables" ξ_i

(SVM-P) $\min_{w,b,\xi} \{ \frac{1}{2}\| w \|^2 + C \sum_{i \in I} \xi_i \,:\, y^i(wx^i - b) \geq 1 - \xi_i \,,\, \xi_i \geq 0 \;\; i \in I \}$

(SVR-P) $\min_{w,b,\xi} \frac{1}{2}\| w \|^2 + C \sum_{i \in I} \xi_i$

$$wx^i - b - y^i - \varepsilon \leq \xi_i \,,\, -wx^i + b + y^i - \varepsilon \leq \xi_i \,,\, \xi_i \geq 0 \;\; i \in I$$

- Corresponding quadratic duals (**check**)

(SVM-D) $\max_\alpha \sum_{i \in I} \alpha_i - \frac{1}{2} \sum_{i \in I} \sum_{j \in I} \alpha_i y^i \langle x^i , x^j \rangle y^j \alpha_j$

$$\sum_{i \in I} y^i \alpha_i = 0$$

$$0 \leq \alpha_i \leq C \qquad i \in I$$

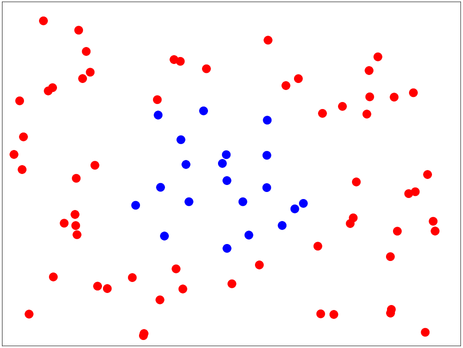
(SVR-D) $\max_\alpha \sum_{i \in I} y^i \alpha_i - \varepsilon \sum_{i \in I} | \alpha_i | - \frac{1}{2} \sum_{i \in I} \sum_{j \in I} \alpha_i \langle x^i , x^j \rangle \alpha_j$

$$\sum_{i \in I} \alpha_i = 0$$

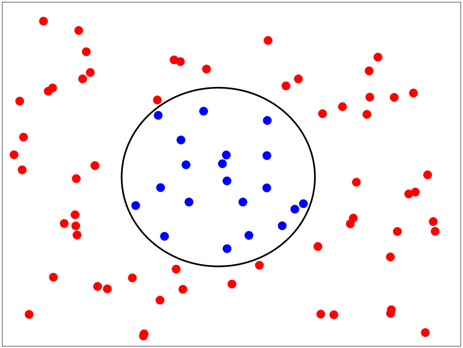
$$-C \leq \alpha_i \leq C \qquad i \in I$$

- Primal-dual relationships: $w^* = \sum_{i \in I} \alpha_i^* [y^i] x^i \implies$ classification / regression of new $\bar{x}$ with $\langle w^* , \bar{x} \rangle - b^* = \sum_{i \in I} \alpha_i^* [y^i] \langle \bar{x} , x^i \rangle - b^*$

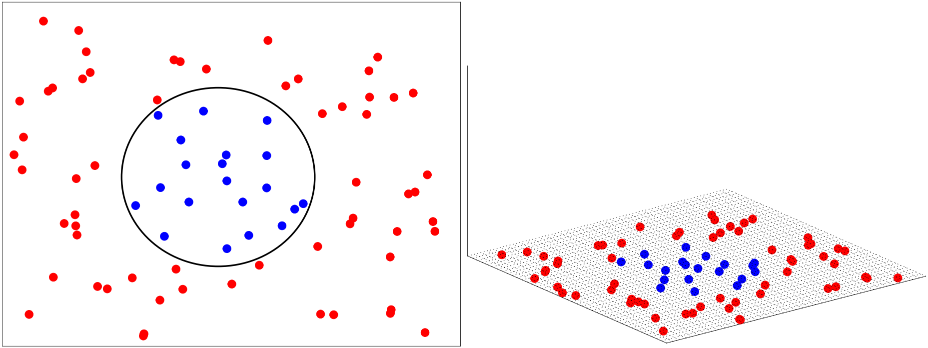
Exercise: prove how to compute w^*, b^* from α^*, discuss why "support vector"



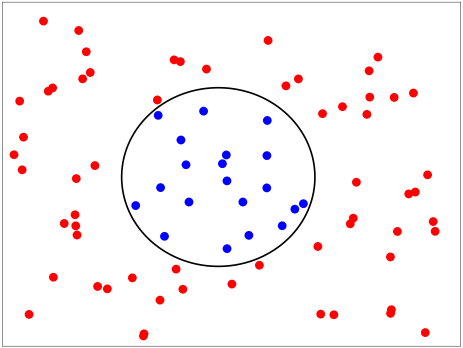
- (Approximate) linear separability

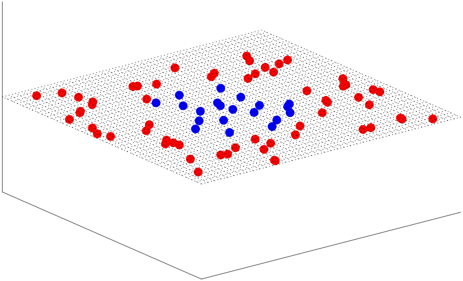


- (Approximate) linear separability rare, (approximate) linear regression weak

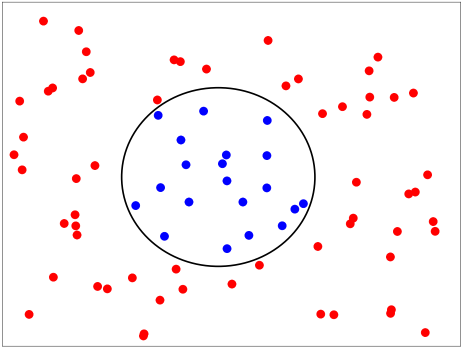


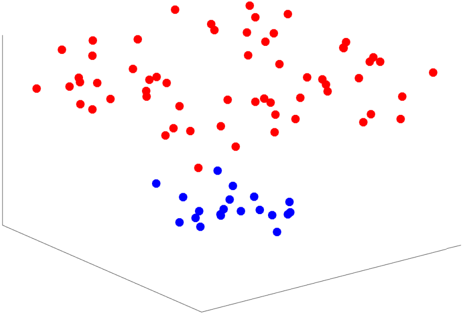
- (Approximate) linear separability rare, (approximate) linear regression weak
- Idea: embed in larger space



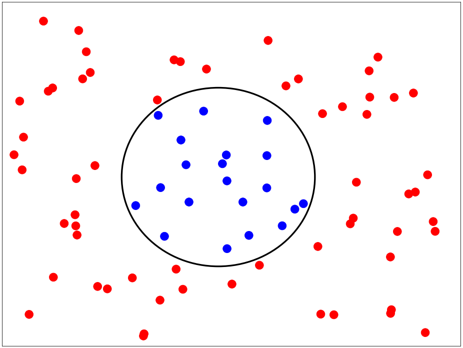


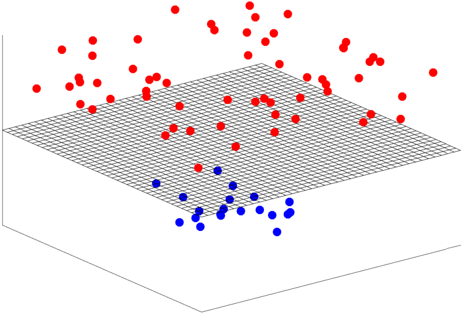
- (Approximate) linear separability rare, (approximate) linear regression weak
- Idea: embed in larger space



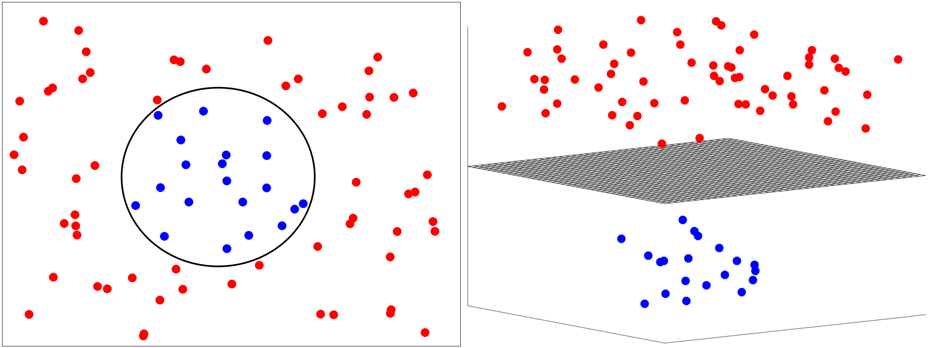


- (Approximate) linear separability rare, (approximate) linear regression weak
- Idea: embed in larger space nonlinearly, then





- (Approximate) linear separability rare, (approximate) linear regression weak
- Idea: embed in larger space nonlinearly, then linear function may work



- (Approximate) linear separability rare, (approximate) linear regression weak
- Idea: embed in larger space nonlinearly, then linear function may work
- Doing this effectivey (how to embed) and efficiently nontrivial

- $\phi : \mathbb{R}^h$ (input space) $\rightarrow \mathcal{F}$ feature space, $x^i \rightarrow \phi(x^i)$
- If $\mathcal{F} = \mathbb{R}^k$ for $k > h$, could just re-do (SVM-P) / (SVR-P) in $\mathbb{R}^k$

- $\phi : \mathbb{R}^h$ (input space) $\rightarrow \mathcal{F}$ feature space, $x^i \rightarrow \phi(\, x^i \,)$
- If $\mathcal{F} = \mathbb{R}^k$ for $k > h$, could just re-do (SVM-P) / (SVR-P) in $\mathbb{R}^k$
- $k \gg h$ good (larger $\equiv$ easier to linearly fit / separate) and bad: fitting cost now scales with k rather than h
- Example: $w \rightarrow W = [\, Q \,,\, q \,]$ and $\langle\, w \,,\, x \,\rangle \rightarrow x^T Q x + q x$ ellipsoidal separation (not really, $Q \succeq 0$ not guaranteed)
- Linear??

- $\phi : \mathbb{R}^h$ (input space) $\rightarrow \mathcal{F}$ feature space, $x^i \rightarrow \phi(x^i)$
- If $\mathcal{F} = \mathbb{R}^k$ for $k > h$, could just re-do (SVM-P) / (SVR-P) in $\mathbb{R}^k$
- $k \gg h$ good (larger $\equiv$ easier to linearly fit / separate) and
  bad: fitting cost now scales with k rather than h
- Example: $w \rightarrow W = [\, Q \, , \, q \,]$ and $\langle\, w \, , \, x \,\rangle \rightarrow x^T Qx + qx$
  ellipsoidal separation (not really, $Q \succeq 0$ not guaranteed)
- Linear?? Indeed: $x \rightarrow F = [\, xx^T \, , \, x \,]$ and $x^T Qx + qx = \langle\, W \, , \, F \,\rangle$
  nonlinearity in mapping ϕ, then linear once in $\mathcal{F}$
- A good thing: nonlinearity on the data (fixed), then problem “easy”

- $\phi : \mathbb{R}^h$ (input space) $\rightarrow \mathcal{F}$ feature space, $x^i \rightarrow \phi(x^i)$
- If $\mathcal{F} = \mathbb{R}^k$ for $k > h$, could just re-do (SVM-P) / (SVR-P) in $\mathbb{R}^k$
- $k \gg h$ good (larger $\equiv$ easier to linearly fit / separate) and
  bad: fitting cost now scales with k rather than h
- Example: $w \rightarrow W = [\, Q \,,\, q \,]$ and $\langle\, w \,,\, x \,\rangle \rightarrow x^T Q x + qx$
  ellipsoidal separation (not really, $Q \succeq 0$ not guaranteed)
- Linear?? Indeed: $x \rightarrow F = [\, xx^T \,,\, x \,]$ and $x^T Q x + qx = \langle\, W \,,\, F \,\rangle$
  nonlinearity in mapping ϕ, then linear once in $\mathcal{F}$
- A good thing: nonlinearity on the data (fixed), then problem "easy"
- Issue: $k \in O(h^2)$, cost grows significantly
- Even worse: $\phi(\cdot) \equiv$ terms of polynomial of degree > 2 (**check**)
- Even worse: one may want $\mathcal{F}$ to be ∞-dimensional

- (SVM/R-D) require kernel function $\kappa(x^i , x^j) = \langle \phi(x^i) , \phi(x^j) \rangle \; \forall \, i , j$
- Classify / interpolate new $\bar{x}$ requires computing $\langle \phi(w^*) , \phi(\bar{x}) \rangle =$
  $= \langle \sum_{i \in I} \alpha^*_i \phi(x^i) , \phi(\bar{x}) \rangle = \sum_{i \in I} \alpha^*_i \langle \phi(x^i) , \phi(\bar{x}) \rangle = \sum_{i \in I} \alpha^*_i \kappa(x^i , \bar{x})$
  whatever ∞-dimensional vector space $\mathcal{F}$ is (general properties of $\langle \cdot , \cdot \rangle$)
  $\Longrightarrow$ can use $\kappa(\cdot, \cdot)$ for everything, no need to ever compute $\phi(\cdot)$
- One κ computation for each support vector x^i s.t. $\alpha^*_i > 0$ (possibly $\ll | \, I \, |$)
- Incredibly clever kernel trick: very large $\mathcal{F}$ s.t. κ is efficient
- κ kernel function for some vector space $\mathcal{F} \iff \int \kappa(x , z) g(x) g(z) dxdz \geq 0$
  $\forall \, g(\cdot)$ s.t. $\int g(x)^2 dx$ is finite (Mercier condition), e.g.
  - Polynomial Kernel (PK): $\kappa(x , z) = (\langle x , z \rangle + 1)^k$ (any k)
  - Gaussian Kernel (GK): $\kappa(x , z) = e^{- \| x - z \|^2 / (2 \sigma^2)}$ (any σ)
  - Sigmoid Kernel (SK): $\kappa(x , z) = \tanh(\sigma \langle x , z \rangle + \delta)$ (some σ, δ and X)
- Many specialised kernels for specific data (trees, graphs, strings, . . .),
  SVR + GK approximates ∞-ly well any f ($\in C^0$, on $[x_- , x_+]$) if $\# X = \infty$

Exercise: discuss why, at least in one dimension, this is not surprising

Outline

- Constrained optimality conditions direct generalization of unconstrained ones
- Constraints $\rightsquigarrow$ some very specific algorithmic issues (will see):
  - Lagrangian multipliers $\equiv$ (possibly, many) "more variables" ($m \gg n$)
  - identifying "the right" $\mathcal{A} \subset \mathcal{I}$, exponential set of candidates
  - (KKT-CS) "very nonlinear" even if everything else (f, g_i, h_j) linear
- Lagrangian multipliers $\rightsquigarrow$ Lagrangian duality: powerful, but max / min
- Convex $\rightsquigarrow$ strong duality, nonconvex $\rightsquigarrow$ relaxation (and ψ "difficult")
- Sometimes "ψ very easy", can do away with $x \implies$ problem only in λ, μ
- Sometimes (D) easier than (P) (e.g., $m \ll n$)
- LP / QP / Conic duality important special cases, easy to use
- Dual information can be extremely useful for algorithms & applications
- Convex $\rightsquigarrow$ algorithms can work in primal space, dual space or both
- Have you said "algorithms"? Yup, let's move on!

[1] M.S. Bazaraa, H.D. Sherali, C.M. Shetty *Nonlinear Programming: Theory and Algorithms*, John Wiley & Sons, 2006

[2] S. Boyd, L. Vandenberghe *Convex Optimization*,
https://web.stanford.edu/~boyd/cvxbook
Cambridge University Press, 2008

[3] A. Frangioni "Solving Semidefinite Quadratic Problems Within Nonsmooth Optimization Algorithms"
http://pages.di.unipi.it/frangio/abstracts.html#COR96
Computers & Operations Research 23(11), 1099–1118, 1996

[4] A. Frangioni "About Lagrangian Methods in Integer Optimization"
http://pages.di.unipi.it/frangio/abstracts.html#AOR05
Annals of Operations Research 139, 163–193, 2005

[5] A. Frangioni ‘Standard Bundle Methods: Untrusted Models and Duality” http://pages.di.unipi.it/frangio/abstracts.html#NDOB in *Numerical Nonsmooth Optimization: State of the Art Algorithms*, A.M. Bagirov, M. Gaudioso, N. Karmitsa, M. Mäkelä, S. Taheri (Eds.), 61–116, Springer, 2020

[6] J.-B. Hiriart-Urruty, C. Lemaréchal *Convex Analysis and Minimization Algorithms* II, Springer-Verlag, 1993

[7] D.G. Luenberger, Y. Ye *Linear and Nonlinear Programming*, Springer International Series in Operations Research & Management Science, 2008

[8] J. Nocedal, S.J. Wright, *Numerical Optimization – second edition*, Springer Series in Operations Research and Financial Engineering, 2006.

[9] W.F. Trench, *Introduction to Real Analysis* http://ramanujan.math.trinity.edu/wtrench/texts/TRENCH_REAL_ANALYSIS.PDF Free Hyperlinked Edition 2.04, December 2013.

[10] CVX: http://cvxr.com

[11] J. Dahl *Conic optimization*
https://docs.mosek.com/slides/2017/aau/conic-opt.pdf, 2017

[12] YALMIP Tutorials https://yalmip.github.io/tutorials

[13] Wikipedia – Universal Approximation Theorem https://en.wikipedia.org/wiki/Universal_approximation_theorem

Outline

- It is obvious that $\mathbb{R}^n = \text{int}(\,\mathbb{R}^n\,)$, hence $\mathbb{R}^n$ is open. Also, there is no x such that $\mathcal{B}(\,x\,,\,r\,) \subseteq \emptyset$, hence $\text{int}(\,\emptyset\,) = \emptyset$, thus $\emptyset$ is also open. Clearly, $S = \mathbb{R}^n \implies \partial S = \emptyset$ since $\nexists\, z \notin S$; similarly, $S = \emptyset \implies \partial S = \emptyset$ since $\nexists\, w \in S$. Thus, for both sets $S = \text{int}(\,S\,)$ and $\partial S = \emptyset$, thus $\text{cl}(\,S\,) = \text{int}(\,S\,) \cup \partial S = S \cup \emptyset = S$, i.e, they are both closed [**back**]

- Trivial: $S = [\,0\,,\,1\,)$. $0 \in \partial S$ and $0 \in S \implies S \neq \text{int}(\,S\,)$ (not open), but $1 \in \partial S \implies \text{cl}(\,S\,) = [\,0\,,\,1\,] \neq S$ (not closed) [**back**]

- No: consider $S_i = (\,-1\,/\,i\,,\,1\,)$. Each S_i is open, but their intersection for $i = 1, \ldots, \infty$ is $S = [\,0\,,\,1\,)$, which is not [**back**]

- No: consider $S_i = [\,0\,,\,1 - 1\,/\,i\,]$. Each S_i is closed, but their union for $i = 1, \ldots, \infty$ is $S = [\,0\,,\,1\,)$, which is not [**back**]

- Take any $d \in T_X(x)$ and the corresponding two sequences $\{ z_i \in X \} \to x$ and $\{ t_i > 0 \} \to 0$ s.t. $d = \lim_{i \to \infty}(z_i - x) / t_i$. For any $\alpha > 0$ set $\bar{t}_i = t_i / \alpha > 0$: clearly $\{ \bar{t}_i \} \to 0$ and $\lim_{i \to \infty}(z_i - x) / \bar{t}_i = \alpha[\lim_{i \to \infty}(z_i - x) / t_i] = \alpha d$ $\implies \alpha d \in T_X(x)$ **[back]**

- Since $\langle \nabla f(x) , d \rangle < 0 \implies d \neq 0$ and $T_X(x)$ is a cone, w.l.o.g. we can assume $\| d \| = 1$. It must be $z_i \neq x$ for all large enough i: in fact, due to $\lim_{i \to \infty} d - (z_i - x) / t_i = 0$, eventually $\| z_i - x \| / t_i \geq \| d \| - \gamma = 1 - \gamma$ however chosen $\gamma > 0$; hence, for $\gamma = 1 / 2$ one has $\| z_i - x \| / t_i \geq 1 / 2 > 0$, that is incompatible with $z_i = x_i \implies \| z_i - x \| = 0$. From now on, all i have to be intended large enough that $z_i \neq x_i$
  This and $f(z_i) - f(x) = \langle \nabla f(x) , z_i - x \rangle + R(z_i - x)$ gives $f(z_i) - f(x) = \| z_i - x \| [\langle \nabla f(x) , (z_i - x) / \| z_i - x \| \rangle + R(z_i - x) / \| z_i - x \|]$. Now, $v_i = (z_i - x) / \| z_i - x \|$ clearly has $\| v_i \| = 1$, and it is "collinear in the limit" with d. Indeed, using again $\lim_{i \to \infty} d - (z_i - x) / t_i = 0$, one has $\lim_{i \to \infty}[\cos(\theta_i) = \langle d , v_i \rangle / (\| d \| \| v_i \|)] = \lim_{i \to \infty} \langle d , z_i - x / \| z_i - x \| \rangle = \lim_{i \to \infty} \langle d , (z_i - x) / t_i \rangle / \| (z_i - x) / t_i \| = \| d \|^2 / \| d \| = 1$. Hence, $\{ v_i \} \to d$ (in the limit they are collinear, and have the same norm).

Hence, $\langle \nabla f(x), v_i \rangle \leq \langle \nabla f(x), d \rangle + \varepsilon$ for large enough i and any $\varepsilon > 0$. Since $\langle \nabla f(x), d \rangle < 0$, take $\varepsilon = -\langle \nabla f(x), d \rangle / 2$ to get that $\exists\, h$ s.t. $\langle \nabla f(x), v_i \rangle \leq -\varepsilon / 2 < 0 \; \forall\, i \geq h$.
Thus, in the sum $r_i = \langle \nabla f(x), v_i \rangle + R(z_i - x) / \| z_i - x \|$], the first term is eventually $\leq -\varepsilon / 2 < 0$. But due to the property of the remainder term $R(\cdot)$, the second term $\to 0$ as $i \to \infty$ ($\Longrightarrow \| z_i - x \| \to 0$). Hence, eventually $r_i \leq -\varepsilon / 4 < 0$. This finally proves that for all large enough i, $f(z_i) - f(x) = \| z_i - x \| r_i < 0$ (recall that eventually $z_i \neq x$) **[back]**

- f convex $\equiv$ $f(\alpha x + (1 - \alpha) z) \leq \alpha f(x) + (1 - \alpha) f(z)$; thus, $(v, x) \in \mathrm{epi}(f) \equiv v \geq f(x)$ and $(w, z) \in \mathrm{epi}(f) \equiv w \geq f(z) \Longrightarrow$ $\alpha v + (1 - \alpha) w \geq \alpha f(x) + (1 - \alpha) f(z) \geq f(\alpha x + (1 - \alpha) z)$, i.e., $\alpha(v, x) + (1 - \alpha)(w, z) \in \mathrm{epi}(f)$, i.e., $\mathrm{epi}(f)$ convex
  For $\Longleftarrow$, $f(x) = \min\{ v : (v, x) \in \mathrm{epi}(f) \}$, thus $\mathrm{epi}(f)$ convex $\Longrightarrow$ $\alpha(f(x), x) + (1 - \alpha)(f(z), z) \in \mathrm{epi}(f) \Longrightarrow f(\alpha x + (1 - \alpha) z) =$ $= \min\{ v : (v, \alpha x + (1 - \alpha) z) \in \mathrm{epi}(f) \} \leq \alpha f(x) + (1 - \alpha) f(z)$ **[back]**

- f convex $\equiv f(\alpha x + (1-\alpha)z) \le \alpha f(x) + (1-\alpha)f(z)$; thus, $f(x) \le v$ and $f(z) \le v \implies f(\alpha x + (1-\alpha)z) \le \alpha f(x) + (1-\alpha)f(z) \le \alpha v + (1-\alpha)v = v$, i.e., $x \in S(f, v)$ and $z \in S(f, v) \implies \alpha x + (1-\alpha)z \in S(f, v) \equiv S(f, v)$ convex
  For $\not\Longleftarrow$, f s.t. $S(f, v)$ convex $\forall\, v$ is quasi-convex. We have already seen in the univariate case that $\exists$ quasi-convex functions that are not convex [**back**]

- "from prime principles". $x \ge 0$ and $\alpha \ge 0 \implies \alpha x \ge 0$, i.e., $\mathbb{R}^n_+$ is a cone; furthermore, $x \ge 0$ and $z \ge 0 \implies \alpha x + (1-\alpha)z \ge 0$, i.e., $\mathbb{R}^n_+$ is convex
  Alternatively, $\mathbb{R}^n_+ = cone(\{ u^1 , \ldots , u^n \})$, u^i for $i = 1, \ldots, n$ being the canonical basis of $\mathbb{R}^n$, as it is immediate to verify [**back**]

- Consider $C = \{ x_1 x_2 \ge 0 : (x_1, x_2) \in \mathbb{R}^2 \}$. It is easy to verify that $C = \mathbb{R}^2_+ \cup \mathbb{R}^2_-$, i.e, it is the union of the positive and negative hortants in $\mathbb{R}^2$. Obviously $x_1 x_2 \ge 0 \implies (\alpha x_1)(\alpha x_2) = \alpha^2 x_1 x_2 \ge 0$, hence C is a cone. But C is not convex: in fact, $(0, 2) \in C$ and $(-2, 0) \in C$, but $\frac{1}{2}(0, 2) + \frac{1}{2}(-2, 0) = (-1, 1) \notin C$, as $-1 \cdot 1 = -1 < 0$ [**back**]

- For i., $x \in \underline{C}$ and $z \in \underline{C}$ imply that $x \in C_i$ and $z \in C_i \; \forall \, i$, which implies that $\alpha z + (1 - \alpha)w \in C_i \; \forall \, i$ (as they all are convex) and therefore $\in \underline{C}$
  Affine mappings are "very convenient" for convexity, because $A[\, \alpha z + (1 - \alpha)w \,] + b = \alpha Az + (1 - \alpha)Aw + [\, \alpha b + (1 - \alpha)b \,] = \alpha[\, Az + b \,] + (1 - \alpha)[\, Aw + b \,]$. That is, even if an affine function is not linear, when making a convex combination (as opposed to a generic linear one) the affine term b can be conveniently managed so that an affine mapping behaves as a linear one under convex combinations. This immediately proves convexity for iii., i.e., $z \in C$ and $w \in C \implies \alpha z + (1 - \alpha)w \in C$, plus $Az + b \in A(\, C \,)$ and $Az + b \in A(\, C \,) \implies \alpha[\, Az + b \,] + (1 - \alpha)[\, Aw + b \,] = A[\, \alpha z + (1 - \alpha)w \,] + b \in A(\, C \,)$, i.e., $A(\, C \,)$ is convex
  iv. is analogous: $x \in C$ and $z \in C \implies \alpha x + (1 - \alpha)w \in C$ and $Ax + b \in A^{-1}(\, C \,)$, $Az + b \in A^{-1}(\, C \,)$, hence $\alpha[\, Ax + b \,] + (1 - \alpha)[\, Az + b \,] = = A[\, \alpha x + (1 - \alpha)z \,] + b \in A^{-1}(\, C \,)$, i.e., $A^{-1}(\, C \,)$ is convex
  ii. and iii. immediately imply v. since αx is a linear mapping and $x = z + w$ is a linear mapping from $\mathbb{R}^{n+n} \rightarrow \mathbb{R}^n$ **[back]**

- $d \in F_X(x) \equiv x + \bar{\varepsilon} d \in X \equiv x + [\bar{\varepsilon} / \alpha](\alpha d) \in X \equiv \alpha d \in F_X(x)$ however chosen $\alpha > 0$, hence $F_X(x)$ is a cone Take any $z \in X$: $x + (z - x) \in X$, hence $d = (z - x) \in F_X(x)$ (with $\bar{\varepsilon} = 1$), hence $X \subseteq x + F_X(x)$ [**back**]

- For $d \in F_X(x)$, X convex $\implies \exists\, \bar{\varepsilon} > 0$ s.t. $x + \varepsilon d \in X \;\forall\, \varepsilon \in [0, \bar{\varepsilon}]$. Thus, for any $\{ \bar{\varepsilon} \geq t_i > 0 \} \to 0$ define $z_i = x + t_i d$: clearly $z_i \in X$, $\{ z_i \} \to x$ and $(z_i - x) / t_i = d \;\forall\, i$, hence $d \in F_X(x)$
  To see that $F_X(x) \neq T_X(x)$ can happen, even with convex X, consider $X = \mathcal{B}_2(0, 1) \subset \mathbb{R}^2$ and $x = [1, 0]$. Obviously, $F_X(x)$ contains all and only the directions $d = [d_1, d_2]$ s.t. $d_1 < 0$, but in addition to all those $T_X(x)$ also contains all the directions $d = [0, d_2]$, characterising the frontier of the set. That is, $F_X(x)$ is an open set and $T_X(x)$ is its closure. This clearly has to do with the fact that ∂X is "smooth" around x: in fact, for $X = \mathcal{B}_1(0, 1)$ one rather has $F_X(x) = \text{cone}(\{ [-1, 1], [-1, -1] \}) = T_X(x)$
  Hence, $X \subseteq x + F_X(x)$ and $F_X(x) \subseteq T_X(x) \implies X \subseteq x + T_X(x)$ [**back**]

- Consider the nonconvex set $X = \{ 0 , 1 \}$ and $0 = x \in X$. $T_X(x) = \{ 0 \}$: in fact, the only way to take a sequence $\{ z_i \} \subset X$ s.t. $\{ z_i \} \to 0$ is to have $z_i = 0$ (eventually), so that t_i is irrelevant and $d = \lim_{i \to \infty} (z_i - x) / t_i = 0$. On the other hand, $d = 1 \in F_X(x)$ ($0 + \bar{\varepsilon} 1 = 1 \in X$ for $\bar{\varepsilon} = 1$), which means that $F_X(x) = \mathbb{R}_+ \supset T_X(x)$ [**back**]

- Assume by contradiction that $\langle \nabla f(x_*) , d \rangle \geq 0 \; \forall \, d \in T_X(x_*)$ but x_* is not optimum: $\exists \, \bar{x} \in X$ s.t. $f(\bar{x}) < f(x_*)$. Since f is convex on X, and both $\bar{x}$ and x_* belong to X, one has $0 > f(\bar{x}) - f(x_*) \geq \langle \nabla f(x_*) , \bar{x} - x_* \rangle$. But $\bar{x} \in X \implies \bar{x} - x_* = d \in F_X(x_*)$, and since X is convex $F_X(x) \subseteq T_X(x)$, hence $d \in T_X(x)$ and therefore $\langle \nabla f(x_*) , d \rangle \geq 0$, yielding the contradiction
If $f \notin C^1$, the contradiction would be found in the same way as long as $\langle g , d \rangle \geq 0$ for any $g \in \partial f(x_*)$ and some $d \in T_X(x_*)$. Hence, for $f \notin C^1$ (TCC) reads $\forall \, d \in T_X(x_*) \, \exists \, g \in \partial f(x_*)$ s.t. $\langle g , d \rangle \geq 0$, or, equivalently, $\max\{ \langle g , d \rangle \, : \, g \in \partial f(x_*) \} \geq 0$. This is consistent with the fact that, for any x, $\frac{\partial f}{\partial d}(x) \geq \langle g , d \rangle \; \forall g \in \partial f(x)$. In order for x_* to be a (local $\equiv$ global) optimum, all feasible directions d must not be of descent. But for d not to be of descent it is enough that $\langle g , d \rangle \geq 0$ for any subgradient g in x_* [**back**]

- Pick any $d \in \mathbb{R}^n$: by definition of $x \in int(X)$ one has $\mathcal{B}(x, \varepsilon) \subset X$, i.e., $x + \varepsilon d \in X$, i.e., $d \in F_X(x)$. But in fact $x + \alpha d \in X \; \forall \alpha \in [0, \varepsilon]$, and therefore $d \in T_X(x)$ without any need for X to be convex: $\mathcal{B}(x, \varepsilon)$ is convex, and convexity is clearly only needed "close to x" for the definition of a "local" object such as $T_X(x)$ [**back**]

- As we know well, $\min\{ \langle \nabla f(x), d \rangle : d \in \mathbb{R}^n \} = \langle \nabla f(x), -\nabla f(x) \rangle = = -\| \nabla f(x) \|^2 \leq 0$. If $\langle \nabla f(x), d \rangle \geq 0 \; \forall d$ then the minimum must be ≥ 0 and therefore $= 0$, i.e., $\| \nabla f(x) \| = 0 \equiv \nabla f(x) = 0$ [**back**]

- Again: $d \in F_X(x) \equiv Ad = 0$, hence $\nabla f(x) = \mu A \implies \langle \nabla f(x), d \rangle = \langle \mu A, d \rangle = \langle \mu, Ad \rangle = 0 \implies$ (TCC) $\implies x$ global optimum since both f and X are convex [**back**]

- $Dx_N + d = \begin{bmatrix} -A_B^{-1}A_N \\ I \end{bmatrix} x_N + \begin{bmatrix} A_B^{-1}b \\ 0 \end{bmatrix} = \begin{bmatrix} A_B^{-1}(b - A_N x_N) \\ x_N \end{bmatrix} = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$
  $AD = [\, A_B \,,\, A_N \,] \begin{bmatrix} -A_B^{-1}A_N \\ I \end{bmatrix} = A_B(-A_B^{-1}A_N) + A_N = -A_N + A_N = 0$
  [**back**]

- Assume that the rows of A are not linearly independent: for A_1 the first row and $\bar{A}$ all the rest, $\exists\,\gamma$ s.t. $A_1 = \gamma^T\bar{A}$. Let b_1 be the first element of b and $\bar{b}$ all the rest: if $b_1 = \gamma^T\bar{b}$, then the first equation is a linear combination of all the remaining ones and therefore irrelevant, in that $\forall\, x$ s.t. $\bar{A}x = \bar{b}$ one has $A_1 x = \gamma^T\bar{A}x = \gamma^T\bar{b} = b_1$. Hence, every solution of the restricted system $\bar{A}x = \bar{b}$ is a solution of the original one (and, obviously, vice-versa), and the first equation can be discarded; repeating the process if necessary eventually leaves with a (reduced) A whose rows are linearly independent. If $b_1 \neq \gamma^T\bar{b}$ instead, the original system has no solution. In fact, for any x s.t. $\bar{A}x = \bar{b}$ one has $A_1 x = \gamma^T\bar{A}x = \gamma^T\bar{b} \neq b_1$, i.e., it is impossible to satisfy all the equations at the same time. Hence the problem is provably empty and there is no point in trying to determine a(n optimal) solution [**back**]

- From the first constraint we get $z = 1 - x$. Plugging this into the second constraint we get $x + w - (1 - x) = 2 \ \equiv\ w = 3 - 2x$. Hence, (R) is $\min\{\, 2x^2 + (3 - 2x)^2 + (1 - x)^2 \,\} = \min\{\, r(\,x\,) = 7x^2 - 14x + 10 \,\}$. Imposing $r'(\,x\,) = 14x - 14 = 0$ gives $x = 1$, whence $z = 0$ and $w = 1$.
  To verify the correctness of the result we write $\nabla f(\,1\,,\,1\,,\,0\,) = [\,4\,,\,2\,,\,0\,]$ and $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$: it is then immediate to verify that $\mu = [\,2\,,\,2\,]$ satisfies $\mu A = \nabla f(\,1\,,\,1\,,\,0\,)$, hence $[\,1\,,\,1\,,\,0\,]$ is optimal [**back**]

- We prove that $x \in \partial S(\,g_i\,,\,0\,) \implies g_i(\,x\,) = 0$ by contradiction: assume $x \in \partial S(\,g_i\,,\,0\,)$ but $g_i(\,x\,) = -\varepsilon < 0$ (the case $g_i(\,x\,) > 0$ is analogous): since $g_i \in C^0\ \exists\,\delta > 0$ s.t. $g_i(\,z\,) \leq g_i(\,x\,) + \varepsilon\,/\,2 = -\varepsilon\,/\,2 < 0\ \forall\, z \in \mathcal{B}(\,x\,,\,\delta\,)$, i.e., $\mathcal{B}(\,x\,,\,\delta\,) \subseteq S(\,g_i\,,\,0\,)$, i.e., $x \in \mathrm{int}(\,S(\,g_i\,,\,0\,)\,)$, contradicting $x \in \partial S(\,g_i\,,\,0\,)$
  The converse implication is not true. Consider the ("ReLU") function $g(\,x\,) = \max\{\,x\,,\,0\,\}$. Clearly, $g \in C^0$, $S(\,g\,,\,0\,) = (\,-\infty\,,\,0\,]$ and $\partial S(\,g\,,\,0\,) = \{\,0\,\}$, but (say) $g(\,-1\,) = 0$ although $-1 \notin S(\,g\,,\,0\,)$. In order for $g_i(\,x\,) = 0$ to be a "good proxy" of $x \in \partial S(\,g_i\,,\,0\,)$, $L(\,g\,,\,0\,)$ must be "thin", i.e., not-full-dimensional. This is hardly a problem in practice, as we can freely

choose our constraint functions. For instance, $h(x) = x$ is such that $S(h, 0) = (-\infty, 0] = S(g, 0)$, but $L(h, 0) = \{0\}$ is "properly thin" [**back**]

- It is obvious that $X = \{[0, 0]\}$: $x_1^2 + (x_2 - 1)^2 - 1 \leq 0 \equiv (x_2 - 1)^2 \leq 1 - x_1^2 \leq 1 \implies x_2^2 - 2x_2 + 1 \leq 1 \equiv x_2(x_2 - 2) \leq 0 \equiv x_2 \in [0, 2]$. Coupled with $x_2 \leq 0$ this gives $x_2 = 0$, and therefore $x_1^2 + (0-1)^2 - 1 \leq 0 \equiv x_1^2 \leq 0 \equiv x_1 = 0$. For $g_1(x_1, x_2) = x_1^2 + (x_2 - 1)^2 - 1$, $\nabla g_1(x_1, x_2) = [2x_1, 2x_2 - 2]^T$, thus $\nabla g_1(0, 0) = [0, -2]^T$. For $g_2(x_1, x_2) = x_2$, $\nabla g_1(x_1, x_2) = [0, 1]^T$. Thus, $d = [d_1, d_2] \in D_X([0, 0])$ requires both $\langle [d_1, d_2], [0, -2] \rangle \leq 0 \equiv -2d_2 \leq 0$ and $\langle [d_1, d_2], [0, 1] \rangle \leq 0 \equiv d_2 \leq 0$, i.e., $d_2 = 0$ while nothing is required on d_1, hence $D_X([0, 0])$ is the x_1 axis as in the picture. On the other hand, since $X = \{[0, 0]\}$, necessarily $T_X(x) = \{[0, 0]\}$ as we have seen already [**back**]

- With $E = JH(x)$, the definition of $D_X(x)$ must also include the equality constraints $Ed = 0$; however, these can be represented as the pair of inequality constraints $Ed \leq 0$, $(-E)d \leq 0$, that are then assumed to be a part of A in the statement of the Lemma to simplify the notation [**back**]

- $c \in \mathcal{C}^* \;\equiv\; c = \lambda^T A$ for some $\lambda \geq 0$, $d \in C \;\equiv\; Ad \leq 0$, hence $\langle c , d \rangle = \langle \lambda^T A , d \rangle = \langle \lambda , v \rangle \leq 0$ since $v = Ad \leq 0$ and $\lambda \geq 0$ [**back**]

- The general separation result states that given a convex set X and a point $\bar{x}$, $\bar{x} \notin X \iff \exists$ a hyperplane $\langle d , x \rangle = \delta$ that separates $\bar{x}$ from X in the sense that $\langle d , x \rangle \leq \delta \; \forall x \in X$ (the whole of X fits in the half-space defined by the hyperplane "in the opposite direction of d") while $\langle d , \bar{x} \rangle > \delta$ ($\bar{x}$ lies in the other half-space defined by the hyperplane, that "in the same direction as d"). In the case of Farkas' lemma, $X = C^*$ and $\bar{x} = c$. Indeed, the lemma states that either $c \in C^*$, or $\exists$ an hyperplane $\langle d , x \rangle = 0$ that separates c from C^*, i.e., such that $\langle d , c \rangle > 0$ and $d \in C \implies \langle d , x \rangle \leq 0 \; \forall x \in C^*$ (as we have already proven). That is, the separating hyperplane in this case is an element of C (and $\delta = 0$). Note that the latter $\implies$ is in fact a $\iff$, since clearly each single $A_i \in C^*$ (take $\lambda \geq 0$ s.t. $\lambda_i = 1$ while $\lambda_j = 0$ for $j \neq i$), hence $\langle\, d\,,\, x\,\rangle \leq 0 \;\forall\, x \in C^* \implies \langle\, d\,,\, A_i\,\rangle \leq 0 \;\forall\, i \;\equiv\; Ad \leq 0 \;\equiv\; d \in C$ **[back]**

- The (TCC) written for $D_X(\,x\,)$ requires that $\langle\, \nabla f(\,x\,)\,,\, d\,\rangle \geq 0 \;\forall\, d \in D_X(\,x\,) = = \{\, d \in \mathbb{R}^n \,:\, Ad \leq 0\,\}$ for properly defined A. The opposite of this condition is that $\exists\, d$ s.t. $Ad \leq 0$ and $\langle\, \nabla f(\,x\,)\,,\, d\,\rangle < 0$. To bring the latter in the right form for applying Farkas' lemma we need to choose $c = -\nabla f(\,x\,)$, so that the condition becomes $\langle\, c\,,\, d\,\rangle > 0$. That not being true (and therefore (TCC) being verified) thus requires $-\nabla f(\,x\,) = c = \lambda^T A$, i.e., $\nabla f(\,x\,) + \lambda^T A = 0$
As previously recalled, the matrix A in the definition of $D_X(\,x\,)$ must also include the equality constraints $Ed = 0$, with $E = JH(\,x\,)$. These are represented as the pair of inequality constraints $Ed \leq 0$, $(-E)d \leq 0$, part of the system $Ad \leq 0$. Thus, they have two separate (vectors of) multipliers $\lambda_+ \geq 0$ and $\lambda_- \geq 0$ (obviously, of the same size), parts of the overall vector λ. The corresponding terms in $\lambda^T A$ then look like "$\ldots \lambda_+^T E + \lambda_-^T(-E)$", i.e., $(\,\lambda_+ - \lambda_-)^T E$. Thus, one can just define $\mu = \lambda^+ - \lambda^-$ and consider a single term $\mu^T E$, except that now the sign of μ is undetermined, while of course expunging λ^+ and λ^- from λ, that now only contains the multipliers of the "true inequality" constraints **[back]**

- In (GC), the multipliers λ_i are only defined for the active constraints, i.e., if $g_i(x) = 0$. This is of course only an issue for inequality constraints, since equality constraints are always active (at feasible points) by definition. Yet, it would be more convenient if the vector of multipliers was always of the same size irrespective of the point x that is being considered. This is indeed possible by always defining a multiplier for each constraint, be it active or not, and then adding the logical condition "if the constraint is not active, the multiplier is 0". This satisfies $\lambda_i \geq 0$ while making the term $\lambda_i \nabla g_i(x)$ in (KKT-G) to vanish, and therefore renders (KKT-G) equivalent to (GC). In a feasible x (satisfying (KKT-F)) one has $-g_i(x) \geq 0 \; \forall \, i \in \mathcal{I}$; since $\lambda_i \geq 0 \; \forall \, i \in \mathcal{I}$, this implies $\sum_{i \in \mathcal{I}} \lambda_i [-g_i(x)] \geq 0$, since all the terms of the sum are ≥ 0; thus, in order for (KKT-CS) to be satisfied, they must necessarily be all 0. This proves that (KKT-CS) (together with (KKT-F)) implies $\lambda_i g_i(x) = 0 \; \forall \, i \in \mathcal{I}$, which in turn proves $g_i(x) < 0 \implies \lambda_i = 0$. Hence, (KKT-CS) (together with (KKT-F)) guarantees that gradients of non-active constraints "disappear from (KKT-G)", thereby making it equivalent to (GC) [**back**]

- Clearly, $C(\,x\,;\,\lambda\,,\,\mu\,) = \mathbb{R}^n$; there are no constraints, hence λ and μ are not even defined, nor are the linear equality and inequality constraints in the definition of $C(\,x\,;\,\lambda\,,\,\mu\,)$. Thus, "$\nabla^2 L(\,x\,;\,\lambda\,,\,\mu\,) \succeq 0$ on the critical cone" is just "$\nabla^2 L(\,x\,) \succeq 0$" as in the ordinary second-order optimality conditions for unconstrained optimization [**back**]

- By dint of having less constraints (in fact, none), the feasible region of $(R_{\lambda,\mu})$ is not smaller than that of (P) (in fact, it being the whole of $\mathbb{R}^n$ it can hardly be larger), i.e., i) holds. In $f(\,x\,) + \langle\,\lambda\,,\,G(\,x\,)\,\rangle + \langle\,\mu\,,\,H(\,x\,)\,\rangle$, the objective of $(R_{\lambda,\mu})$, if x is feasible then $H(\,x\,) = 0 \implies \langle\,\mu\,,\,H(\,x\,)\,\rangle = 0$, and $G(\,x\,) \leq 0 \implies \langle\,\lambda\,,\,G(\,x\,)\,\rangle \leq 0$ (since $\lambda \geq 0$); thus, $f(\,x\,) + \langle\,\lambda\,,\,G(\,x\,)\,\rangle + \langle\,\mu\,,\,H(\,x\,)\,\rangle \leq f(\,x\,)$, i.e., ii) holds. Hence, $(R_{\lambda,\mu})$ is a relaxation of (P) [**back**]

- For any $x \in \mathbb{R}^n$, $l_x(\lambda, \mu) = f(x) + \langle \lambda, G(x) \rangle + \langle \mu, H(x) \rangle$ is a linear function in λ and μ: in fact, since x is fixed, $f(x)$ is a fixed number and $G(x)$, $H(x)$ are fixed vectors. In other words, "all the nonlinearity of the function is related to x": when x is fixed, the function is linear in the other variables λ and μ. Thus, $\psi(\lambda, \mu)$ is the pointwise minimum of all the (uncountably ∞-ly many) linear functions $l_x(\cdot)$, one for each $x \in \mathbb{R}^n$, and therefore concave since the pointwise maximum of convex functions is convex: $\psi(\lambda, \mu) = \max\{ l_x(\lambda, \mu) : x \in \mathbb{R}^n\} \equiv -\min\{ -l_x(\lambda, \mu) : x \in \mathbb{R}^n \}$, $l_x(\cdot)$ are linear thus $-l_x(\cdot)$ are linear and therefore convex (as well as concave), hence $\psi(\cdot)$ is the opposite of a convex function and thus concave **[back]**

- The obvious condition is $L(x; \lambda, \mu)$ strictly ($\Longleftarrow$ strongly) convex on x, which also means that $(R_{\lambda,\mu})$ is "easy" (can be solved by local methods). This implies h_i affine, g_i convex and at least one among f and the g_i strictly ($\Longleftarrow$ strongly) convex. In our applications the g_i are invariably affine, hence the condition becomes f strictly ($\Longleftarrow$ strongly) convex **[back]**

- For $L(x\,;\lambda\,,\mu) = f(x) + \langle\,\lambda\,,\,G(x)\,\rangle + \langle\,\mu\,,\,H(x)\,\rangle$, $\nabla_x L(x\,;\lambda\,,\mu) =$ $= \nabla f(x) + \sum_{i\in\mathcal{I}} \lambda_i \nabla g_i(x) + \sum_{j\in\mathcal{J}} \mu_j \nabla h_j(x)$; thus $\nabla_x L(x_*\,;\lambda_*\,,\mu_*) = 0$ is precisely (KKT-G), i.e., x_* is a stationary point for $L(\,\cdot\,;\lambda_*\,,\mu_*)$, which is convex because (P) is convex, i.e., $f(\cdot)$ is convex, each $g_i(\cdot)$ is convex and therefore $\langle\,\lambda_*\,,\,G(\cdot)\,\rangle$ is (since $\lambda_* \geq 0$), and $H(\cdot)$ is affine and therefore $\langle\,\mu_*\,,\,H(\cdot)\,\rangle$ is (affine, hence convex, irrespectively on the sign of μ_*). Hence x_* is a minimum of $L(\,\cdot\,;\lambda_*\,,\mu_*)$, and therefore optimal for (R_{λ_*,μ_*}): this proves $v(D) \geq \psi(\lambda_*\,,\mu_*) = L(x_*\,;\lambda_*\,,\mu_*) = f(x_*) + \langle\,\lambda_*\,,\,G(x_*)\,\rangle + \langle\,\mu_*\,,\,H(x_*)\,\rangle$. But x_* is optimal for (P) hence feasible, therefore $H(x_*) = 0$; furthermore, $\langle\,\lambda_*\,,\,G(x_*)\,\rangle = 0$ by (KKT-CS). This finally yields $L(x_*\,;\lambda_*\,,\mu_*) = f(x_*) =$ $= v(P) \geq v(D)$, finishing the proof [**back**]

- For $(P_r) \equiv \min\{ f(x) : G(x) - r \leq 0 \}$, consider the Lagrangian function $L_r(x; \lambda) = f(x) + \langle \lambda, G(x) - r \rangle = L(x; \lambda) - \langle \lambda, r \rangle$. Let x_* and λ_* be optimal for (P_0) and its dual (D_0), respectively: we know that x_* is also optimal for the Lagrangian relaxation $(R_{\lambda_*}) \equiv \min_x\{ L(x; \lambda_*) : x \in \mathbb{R}^n \}$. But $L(x \lambda)$ and $L_r(x \lambda)$ only differ for the term $-\langle \lambda, r \rangle$, that does not depend on x: thus, x_* is also the optimal solution of the Lagrangian relaxation $(R_{r,\lambda_*}) \equiv \min_x\{ L_r(x; \lambda_*) : x \in \mathbb{R}^n \}$ of (P_r). Hence, $v(P_r) \geq v(R_{r,\lambda_*}) =$ $= L_r(x_*; \lambda_*) = L(x_*; \lambda_*) - \langle \lambda_*, r \rangle = v(P_0) - \langle \lambda_*, r \rangle$ **[back]**

- As it can be expected, it is (P). In fact, rewrite (D) in the same form as (P), i.e., $(D) = (\bar{P})\ -\min\{ \bar{c}\lambda : \bar{A}\lambda \geq \bar{b} \}$ with $\bar{c} = -b$, $\bar{A} = [\, A, -A, I \,]^T$ and $\bar{b} = [\, c^T, -c^T, 0 \,]^T$: its dual is $(\bar{D})\ -\max\{ w\bar{b} : w\bar{A} = \bar{c},\ w \geq 0 \}$. Note that if $A \in \mathbb{R}^{m \times n}$, $\bar{A} \in \mathbb{R}^{2m+n \times m}$ and therefore $w \in \mathbb{R}^{2m+n}$. We write $w = [\, x^-, x^+, s \,]$ ($x^-, x^+ \in \mathbb{R}^n$ and $s \in \mathbb{R}^m$) and plug this into $(\bar{D})$ to get $(\bar{D})\ -\max\{ cx^- - cx^+ : Ax^- - Ax^+ + s = -b,\ x^- \geq 0,\ x^+ \geq 0,\ s \geq 0 \}$. With some easy algebra and $-\max\{\ \} = \min\{ -\ \}$ we transform this into $(\bar{D})\ \min\{ c(x^+ - x^-) : A(x^+ - x^-) - s = b,\ x^- \geq 0,\ x^+ \geq 0,\ s \geq 0 \}$. We now substitute the pair of variables x^+ and x^-, both constrained in sign, with

$x = x^+ - x^-$ that is not. We then note that, since $s \geq 0$ and s does not appear in the objective (has 0 coefficients there), $Ax - s = b$ is equivalent to $Ax \geq b$. Hence, $(\bar{D})$ $\min\{\, cx \,:\, Ax \geq b \,\} = (P)$: the dual of the dual is the primal [**back**]

- There are two ways develop a dual for a LP in a different form: either one re-develops the Lagrangian relaxation and the closed formula or, like in previous exercise, one rewrites the primal in a form for which the dual is known. For instance, (P) $\min\{\, cx \,:\, Ax \geq b \,,\, x \geq 0 \,\} \equiv \min\{\, cx \,:\, \bar{A}x \geq \bar{b} \,\}$ with $\bar{A}^T = [\, A^T \,,\, I \,]$ and $\bar{b}^T = [\, b^T \,,\, 0 \,]$: plugging this into the (D) formula we already have gives (D) $\max\{\, \bar{\lambda}\bar{b} \,:\, \bar{\lambda}\bar{A} = c \,,\, \bar{\lambda} \geq 0 \,\}$ with $\bar{\lambda}^T = [\, \lambda \,,\, \lambda_+ \,]$. But $\bar{\lambda}\bar{A} = \lambda A + \lambda_+$, and since the λ_+ do not appear (have 0 coefficient) in the objective they are "slack variables" and they can be eliminated by just rewriting the problem as (D) $\max\{\, \lambda b \,:\, \lambda A \leq c \,,\, \lambda \geq 0 \,\}$: sign constraints on the primal variables change the dual constraints from equalities to inequalities This can be generalised by developing a primal-dual correspondence table that allows to directly derive the dual of an LP written in "any" form, i.e., where each constraint can be either an equality or an inequality (of both senses) and each variable can be either constrained in sign (in both ways) or not; w.l.o.g., this can be written

$$\min \begin{bmatrix} c^+ & c^- & c^0 \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ x^0 \end{bmatrix}$$

$$\begin{bmatrix} A^+_+ & A^-_+ & A^0_+ \\ A^+_- & A^-_- & A^0_- \\ A^+_0 & A^-_0 & A^0_0 \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ x^0 \end{bmatrix} \begin{matrix} \leq \\ \geq \\ = \end{matrix} \begin{bmatrix} b_+ \\ b_- \\ b_0 \end{bmatrix} \quad , \quad \begin{matrix} x^+ \geq 0 \\ x^- \leq 0 \end{matrix}$$

Again, the trick is to cook up $\bar{A}$, $\bar{b}$ and $\bar{c}$ that express the same problem written in a form for which we already know (D). This requires the application of a few simple tricks of the trade, such as that the (block of) equality constraint(s) $A_0 x = b_0$ is equivalent to the pair $A_0 x \leq b_0$ and $A_0 x \geq b_0$, and that $A_+ x \leq b_+$ is equivalent to $(-A_+)x \geq (-b_+)$, finally yielding

$$\min \begin{bmatrix} c^+ & c^- & c^0 \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ x^0 \end{bmatrix}$$

$$\begin{bmatrix} -A_+^+ & -A_+^- & -A_+^0 \\ A_-^+ & A_-^- & A_-^0 \\ A_0^+ & A_0^- & A_0^0 \\ -A_0^+ & -A_0^- & -A_0^0 \\ I & 0 & 0 \\ 0 & -I & 0 \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ x^0 \end{bmatrix} \geq \begin{bmatrix} -b_+ \\ b_- \\ b_0 \\ -b_0 \\ 0 \\ 0 \end{bmatrix}$$

whose dual is

$$\max \begin{bmatrix} y_+ & y_- & y_0^+ & y_0^- & s^+ & s^- \end{bmatrix} \begin{bmatrix} -b_+ \\ b_- \\ b_0 \\ -b_0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_+ & y_- & y_0^+ & y_0^- & s^+ & s^- \end{bmatrix} \begin{bmatrix} -A_+^+ & -A_+^- & -A_+^0 \\ A_-^+ & A_-^- & A_-^0 \\ A_0^+ & A_0^- & A_0^0 \\ -A_0^+ & -A_0^- & -A_0^0 \\ I & 0 & 0 \\ 0 & -I & 0 \end{bmatrix} = \begin{bmatrix} c^+ & c^- & c^0 \end{bmatrix}$$

$$\begin{bmatrix} y_+ & y_- & y_0^+ & y_0^- & s^+ & s^- \end{bmatrix} \geq 0$$

We now have to use other simple tricks of the trade, in particular redefining $y_+ = -y_+$ (hence $y_+ \leq 0$) and $y_0 = y_0^+ - y_0^-$ (hence $y_0 \gtrless 0$), plus eliminating the slack variables s^+ and s^- by turning the corresponding constraints into inequalities (with the appropriate verse): this finally yields a dual problem where the data has the "natural size" of the primal (no extra rows/columns required)

$$\max \begin{bmatrix} y_+ & y_- & y_0 \end{bmatrix} \begin{bmatrix} b_+ \\ b_- \\ b_0 \end{bmatrix}$$

$$\begin{bmatrix} y_+ & y_- & y_0 \end{bmatrix} \begin{bmatrix} A_+^+ & A_+^- & A_+^0 \\ A_-^+ & A_-^- & A_-^0 \\ A_0^+ & A_0^- & A_0^0 \end{bmatrix} \quad , \quad \begin{array}{l} y_+ \leq 0 \\ y_- \geq 0 \end{array}$$
$$\begin{matrix} \wedge\!| & \vee\!| & || \end{matrix}$$
$$\begin{bmatrix} c^+ & c^- & c^0 \end{bmatrix}$$

thereby proving the validity of the general primal-dual correspondence table

max	c	b	$A_i x \leq b_i$	$A_i x \geq b_i$	$A_i x = b_i$	$x_j \geq 0$	$x_j \leq 0$	$x_j \gtrless 0$
min	b	c	$y_i \geq 0$	$y_i \leq 0$	$y_i \gtrless 0$	$yA^j \geq c_j$	$yA^j \leq c_j$	$yA^j = c_j$

where, as usual, A_i and A^j are, respectively, the i-th row and the j-th column of the coefficients matrix A **[back]**

- An LP is convex, and clearly regular as (AffC) trivially holds: thus, provided that (P) has an optimal solution x_*, then (D) has an optimal solution λ_* and strong duality holds, i.e., $cx_* = \lambda_* b$, as a consequence of the general result. Since (D) is also an LP and its dual is (P) [see previous exercise], we can apply the result to (D) to prove that $\exists\, \lambda_* \implies \exists\, x_*$: thus, strong duality always holds whenever at least one of the two LPs has an optimal solution. To find cases where strong duality fails we therefore have to require at least one of the problems is empty, but one is not enough. In fact, if (P) is unbounded below, i.e., $v(P) = -\infty$, then by weak duality (D) must be empty (every feasible solution to (D) provides a finite lower bound to $v(P)$), i.e., $v(D) = -\infty$ as well by the definition of the maximum over an empty set. Of course this works symmetrically if (D) is unbounded ((P) must be empty), hence strong duality holds (albeit in a sort of "degenerate" way where optimal values are infinite) in those cases as well. Yet, another case remains: that where both (P) and (D) are empty. This is indeed possible, one example being
$$c = [\,0\,,\,1\,] \;\;,\;\; A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \;\;,\;\; b = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$
i.e., (P) $\min\{\, x_2 \;:\; x_1 \leq -1\,,\, -x_1 \leq -1 \,\}$ and
(D) $\max\{\, -\lambda_1 - \lambda_2\,;\, \lambda_1 - \lambda_2 = 0\,,\, 0 = 1\,,\, \lambda_1 \geq 0\,,\, \lambda_2 \geq 0 \,\}$. It is immediately

obvious that both (P) and (D) are empty, hence $v(P) = +\infty > -\infty = v(D)$, i.e., strong duality "falls spectacularly". This is a rather extreme case, though, that hardly ever occurs in practice. It is tied to the fact that (P) "is at the same time empty and unbounded": it would be unbounded due to the variable x_2 if it were possible to find any feasible value for x_1, which is not [**back**]

- $L(x\,;\,\lambda) = \frac{1}{2}x^T Qx + qx + \lambda(b - Ax) = \lambda b + [\,R_\lambda(x) = \frac{1}{2}x^T Qx + (q - \lambda A)x\,]$. Hence, $\nabla R_\lambda(x) = Qx + q - \lambda A = 0 \equiv x = Q^{-1}(q - \lambda A)$. Rather than directly plugging this into $R_\lambda(x)$, one gets a more compact formula by defining $v = q - \lambda A \equiv x = Q^{-1}v$. Then, $R_\lambda(Q^{-1}v) = \frac{1}{2}v^T Q^{-T} QQ^{-1}v + vQ^{-1}v =$ $= -\frac{1}{2}v^T Q^{-1}v$, which yields the announced (D)
  If the constraints had been $Ax = b$, not much would change except that we would have the unconstrained μ in place of $\lambda \geq 0$ all along, hence (D) would look the same save for the ≥ 0 constraint [**back**]

- This is basically the sum of a QP with strictly convex Q and of an LP. Thus, $L(\,z\,,\,w\,;\,\lambda\,) = \frac{1}{2}z^TQz + qz + pw + \lambda(\,b - Az - Ew\,) = \lambda b + [\,R_\lambda(\,z\,) = \frac{1}{2}z^TQz + (\,q - \lambda A\,)z\,] + [\,R_\lambda(\,w\,) = (\,p - \lambda E\,)w\,]$. As in the LP case, minimizing $R_\lambda(\,w\,)$, which is linear, yields $-\infty$ unless $p - \lambda E = 0$, in which case it yields 0. As in the previous exercise, minimizing $R_\lambda(\,z\,)$ yields $z = Q^{-1}v$ where $v = q - \lambda A$ and $R_\lambda(\,Q^{-1}v\,) = -\frac{1}{2}v^TQ^{-1}v$. All in all,
  $(D)\;\; \max\{\,\lambda b - \frac{1}{2}v^TQ^{-1}v \;:\; \lambda E = p\,,\, \lambda A - v = q\,,\, \lambda \geq 0\,\}$ [**back**]

- The start is identical to the first exercise in the slide: $L(\,x\,;\,\lambda\,) = = \frac{1}{2}x^TQx + qx + \lambda(\,b - Ax\,) = \lambda b + [\,R_\lambda(\,x\,) = \frac{1}{2}x^TQx + (\,q - \lambda A\,)x\,]$. However, $\nabla R_\lambda(\,x\,) = Qx + q - \lambda A = 0$ no longer has a closed formula that allows to do away with x: hence, one has to leave x in the formulation. Yet, the usual trick works for simplifying the objective $\frac{1}{2}x^TQx + (\,q - \lambda A\,)x$: since $q - \lambda A = -Qx$, multiplying by x we obtain $(\,q - \lambda A\,)x = -x^TQx$, and therefore $\frac{1}{2}x^TQx + (\,q - \lambda A\,)x = -\frac{1}{2}x^TQx$. This is crucial on two accounts: first it yields a concave quadratic term in the objective, that is maximised, and second it does away with the bilinear term λAx. All in all, we obtain

  $$(D)\;\; \max\{\,\lambda b - \tfrac{1}{2}x^TQx \;:\; Qx + q - \lambda A = 0\,,\, \lambda \geq 0\,\}$$

Note that the x variables in (D) are formally distinct from those in (P)
If $Q \not\succeq 0$, the above development fails in that $Qx + q - \lambda A = 0$ is no longer equivalent to "x is a minimum of $R_\lambda(x) = \frac{1}{2}x^T Qx + (q - \lambda A)x$". In fact, Q then has directions of negative curvature, along which $R_\lambda(x)$ is unbounded below: hence, $\psi(\lambda) = -\infty \; \forall \lambda \implies v(D) = -\infty$, as we have already seen happening in the example $\min\{ -x^2 \,:\, 0 \leq x \leq 1 \}$. This does not mean that Lagrangian techniques cannot be used in nonconvex problems, far from it: but (very roughly speaking) one have to use partial Lagrangian relaxations where not all the constraints are relaxed, so that $\psi(\lambda) > -\infty$ may happen. We will not be able to delve further into this idea [**back**]

- It is obvious that a SOCP constraint of a vector in $\mathbb{R}^1$ (a single variable) is $x \geq 0$, i.e., a sign constraint: thus, any LP is a SOCP. Also, $|| [x, (t-s)/2] || \leq (t+s)/2 \equiv \sqrt{\|x\|_2^2 + (t-s)^2/4} \leq (t+s)/2 \equiv \|x\|_2^2 + (t-s)^2/4 \leq (t+s)^2/4 \equiv \|x\|_2^2 \leq ts \equiv \|x\|_2^2/s \leq t$ if $s > 0$ (and it actually works if $s = 0$, too, written in the form $\|x\|_2^2 \leq ts$)
  Used with $s = 1$, this proves that one can transform a convex quadratic constraint into a conic one: while $\|x\|_2^2$ is only the simplest of convex

quadratic functions, it can be used to construct any convex quadratic function via an appropriate affine mapping. Indeed, let $Q = RR$: $x^T Qx = \| z \|_2^2$ with $Rx = z$, and affine mappings can always be represented in a SOCP (they are linear constraints) by adding new variables if necessary. Similarly, t can be transformed to any linear form. In fact, this is necessary already to bring the above form to the "standard" SOCP definition, with $[\, x \,,\, w \,,\, z \,]$ the vector of variables (in this order), $w = (\, t - s \,) \,/\, 2$ and $z = (\, t + s \,) \,/\, 2$
Conversely, it is obvious as $x^2 \,/\, s$ is not a standard convex quadratic function. Actually, $x^2 \,/\, s - t \leq 0$ can be written as the quadratic constraint $x^2 - ts \leq 0$, but that quadratic function is easily seen not to be convex: in fact, its Hessian is $Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ whose eigenvalues are 2, 1 and -1 [**back**]

- Let λ_1 and λ_2 be the eigenvalues of any symmetric real 2×2 matrix $Q = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$: it is well-known that $\mathrm{tr}(\, Q\,) = a + b = \lambda_1 + \lambda_2$, while $\det(\, Q\,) = \lambda_1\lambda_2$. It is also obvious that $Q \succeq 0 \implies a \geq 0 \wedge b \geq 0$, for otherwise it is trivial to find $v \in \mathbb{R}^2$ s.t. $v^T Q v < 0$. Thus, $Q \succeq 0 \implies \mathrm{tr}(\, Q\,) = \lambda_1 + \lambda_2 \geq 0$. As the conditions $a \geq 0 \wedge b \geq 0$ are necessary anyway, $Q \succeq 0 \equiv \det(\, Q\,) \geq 0$, since two numbers whose sum is ≥ 0 are both ≥ 0 $\iff$ their product is also ≥ 0. Hence, $Q \succeq 0 \equiv a \geq 0 \wedge b \geq 0 \wedge \det(\, Q\,) = ab - c^2 \geq 0 \equiv ab \geq c^2$: but we have seen in the previous exercise that all these conditions are SOCP-representable, hence any SOCP is a SDP [**back**]

- To apply Lagrangian duality we rewrite $(P)\ \min\{\, cx \,:\, v = Ax - b\,,\ v \geq_K 0\,\}$ and then we consider its partial Lagrangian relaxation w.r.t. the linear constraints only: $(R_\lambda)\ \min\{\, cx + \lambda(\, v + b - Ax\,) \,:\, v \geq_K 0\,\}$. Due to the linearity of the objective, (R_λ) decomposes into two independent problems: $\min\{\, (\, c - \lambda A\,)x \,:\, x \in \mathbb{R}^n\,\}$ and $\min\{\, \lambda v \,:\, v \geq_K 0\,\} \equiv \min\{\, \lambda v \,:\, v \in K\,\}$ (plus the constant term λb). The first is exactly the same as in the LP case and it gives rise to the constraint $\lambda A = c$. The second is a problem with a constraint in a cone, and therefore "rather prone to be unbounded below". In fact, if there exists any $\bar{v} \in K$ s.t. $\lambda\bar{v} < 0$, then the second problem is unbounded below, as by definition of cone $\alpha\bar{v} \in K\ \forall\, \alpha \geq 0$. Thus, the second problem is unbounded below unless there is no such $\bar{v}$: in other words, it must be $\lambda v \geq 0\ \forall\, v \in K$, which by definition corresponds to $\lambda \in K^D$. All in all this gives the announced $(D)\ \max\{\, \lambda b \,:\, \lambda A = c\,,\ \lambda \in K^D\,\}$. It is easy to see how this proof is a very direct generalization of that for LPs [**back**]

- $(PM_{\mathscr{B},\bar{x},\mu})$ $\min\{\, v + \mu\|\, d\,\|^2 / 2 \,:\, uv - Gd \geq -\alpha \,\}$, where G is the matrix having all the g^h as rows, α is the vector having all the α^h as entries, and u is the all-1 vector. This is a QP with "some quadratic and some linear variables" and whose quadratic variables have strictly convex $Q\,[= \mu I]$, as we have seen in a previous exercise: plugging the data of $(PM_{\mathscr{B},\bar{x},\mu})$ in the formula we obtained then gives $\max\{\, \theta(-\alpha) - \frac{1}{2} v^T Q^{-1} v \,:\, \theta u = 1\,,\, \theta(-G) - v = 0\,,\, \theta \geq 0 \,\} \equiv -\min\{\, \frac{1}{2\mu}\|\, \theta G\,\|^2 + \alpha\theta \,:\, \theta \in \Theta \,\}$ with $\Theta = \{\, \theta \geq 0 \,:\, \theta u = 1 \,\}$. The map between the optimal primal and dual solutions comes first from (KKT-G) of the d variables, i.e., $\theta^* G = -\nabla_d f(\, d^*\,) = -\mu d^* \;\equiv\; d^* = -(1 / \mu)[\, \theta^* G\,]$ (note that in the derivation of KKT the constraints are written as $G(\, x\,) \leq 0$, which explains why it is "$\theta^* G$" rather than "$\theta^*(-G)$"). Then, (KKT-CS) gives $\theta^*[\, uv^* - Gd^* + \alpha\,] = 0 \implies v^* = \theta^* G d^* - \alpha\theta^*$ (since $\theta^* u = 1$) $\implies$ $v^* = -(1 / \mu)\|\, \theta^* G\,\|^2 - \alpha\theta^*$ $[\leq 0]$ **[back]**

- The setting is that of using a proximal bundle method to minimize the convex dual function $v(R_\lambda) = \varphi(\lambda) = \max\{ L(x; \lambda) = f(x) + \lambda(Ax - b) : x \in \mathbb{R}^n \}$ of the original problem (P) $\max\{ f(x) : Ax = b, x \in \mathbb{R}^n \}$, with concave f. Here, iterates λ^h produce optimal solutions x^h of (R_{λ^h}) that give $\varphi(\lambda^h)$ in the obvious way and $Ax^h - b = g^h \in \partial\varphi(\lambda^h)$. Thus, for linearization errors w.r.t. the stability centre $\bar{\lambda}$ (and the corresponding optimal solution $\bar{x}$ of $(R_{\bar{\lambda}})$), one has $\alpha^h = \varphi(\bar{\lambda}) - \varphi(\lambda^h) - \langle Ax^h - b, \bar{\lambda} - \lambda^h \rangle =$ $f(\bar{x}) + \bar{\lambda}(A\bar{x} - b) - f(x^h) - \lambda^h(Ax^h - b) - \langle Ax^h - b, \bar{\lambda} - \lambda^h \rangle =$ $f(\bar{x}) + \bar{\lambda}(A\bar{x} - b) - f(x^h) - \bar{\lambda}(Ax^h - b) = L(\bar{x}; \bar{\lambda}) - L(x^h; \bar{\lambda}) \geq 0$. Hence, $\alpha^h = 0 \iff \bar{x}$ and x^h have the same objective value in the Lagrangian relaxation w.r.t. $\bar{\lambda}$, and since $\bar{x}$ is optimal for $(R_{\bar{\lambda}})$ this implies that x^h is optimal as well, as announced. Then, when $\bar{\lambda}$ is proven optimal by $d^* = 0 \equiv z^* = 0 \equiv \theta^* G = 0 \equiv \sum_{h \in \mathscr{B}} g^h \theta^*_h = 0 \equiv \sum_{h \in \mathscr{B}} (Ax^h - b)\theta^*_h = 0 \equiv A \sum_{h \in \mathscr{B}} x^h \theta^*_h = b \sum_{h \in \mathscr{B}} \theta^*_h \equiv Ax^* = b$ for $x_* = \sum_{h \in \mathscr{B}} x^h \theta^*_h$, x^* is a feasible solution of (P) and $\varphi(\bar{\lambda}) \geq L(x^*; \bar{\lambda}) = f(x^*) + \bar{\lambda}(Ax^* - b) = = f(x^*)$. But since f is concave, and $\bar{\lambda}$ is optimal, x^* is also optimal for $(R_{\bar{\lambda}})$: in fact, $\sigma^* = 0 \equiv \alpha^h = 0 \equiv L(\bar{x}; \bar{\lambda}) = L(x^h; \bar{\lambda})$ $\forall\, h$ s.t. $\sigma^*_h > 0$, hence $\varphi(\bar{\lambda}) =$ $= L(\bar{x}; \bar{\lambda}) = \sum_{h \in \mathscr{B}} \theta^*_h L(x^h; \bar{\lambda}) = \sum_{h \in \mathscr{B}} \theta^*_h f(x^h) + \sum_{h \in \mathscr{B}} \theta^*_h \bar{\lambda}(Ax^h - b) =$

$\sum_{h \in \mathscr{B}} \theta^*_h f(\, x^h \,) + \bar{\lambda} \sum_{h \in \mathscr{B}} \theta^*_h (\, A x^h - b \,) = \sum_{h \in \mathscr{B}} \theta^*_h f(\, x^h \,) + \bar{\lambda}(\, A x^* - b \,) =$ $\sum_{h \in \mathscr{B}} \theta^*_h f(\, x^h \,) \leq f(\, x^* \,)$ (by concavity). Thus, $f(\, x^* \,) = \varphi(\, \bar{\lambda} \,) \geq v(D) \geq$ $\geq v(P) \geq f(\, x^* \,)$: x^* is optimal for (P) and $\bar{\lambda}$ is optimal for (D).
The proof actually shows that $d^* = 0 \; \equiv \; z^* = 0$ and $\sigma^* \leq \varepsilon$ means that x^* is ε-optimal, and that $\| \, z^* \, \| \leq \delta \; \equiv \; \| \, A x^* - b \, \| \leq \delta$, i.e., x^* is approximately feasible (which is all that can be required from a numerical algorithm). Since it can be shown that $\sigma^* \to 0$ and $\| \, z^* \, \| \to 0$ as $h \to \infty$, bundle methods are able to (asymptotically) provide optimal solutions to (P). This is also true for convex nonlinear $G(\, x \,) \leq 0$: the fundamental steps are that, due to the constraints $\lambda \geq 0$ in (D), the optimality condition is not on d^* but on its projection over the active constraints, i.e., $\bar{d}_i = 0$ if $\bar{\lambda}_i = 0$ and $d^*_i < 0$ while $\bar{d}_i = d^*_i$ otherwise, and that this implies $G(\, x_* \,) \leq 0$ (via convexity) [**back**]

- (SVM-P) is $\min\{\, Cu\xi + \|\, w \,\|^2 / 2 \,:\, \xi + YXw - yb \geq u \,,\, \xi \geq 0 \,\}$, where X is the matrix having the x^i as rows, y is the vector having the y^i as entries, $Y = \operatorname{diag}(\, y \,)$ and u is the all-1 vector. This is again a QP with "some linear-only variables" and whose "quadratic variables" have strictly convex $Q\,[= I]$: plugging the data of (SVM-P) in the formula we obtained in the previous exercise yields
$\max\{\, \alpha u - \|\, v \,\|^2 / 2 \,:\, \alpha + s = Cu \,,\, \alpha y = 0 \,,\, \alpha YX - v = 0 \,,\, \alpha \geq 0 \,,\, s \geq 0 \,\}$
The dual variables s of the $\xi \geq 0$ constraints have no cost, i.e., they are slack variables and can be eliminated by changing the first constraints to $\alpha \leq Cu$. This yields the desired $\max\{\, \alpha u - \alpha^T Y (X^T X) Y \alpha / 2 \,:\, \alpha y = 0 \,,\, 0 \leq \alpha \leq Cu \,\}$ by just substituting away v. With the same notation, (SVR-P) is
$\min\{\, Cu\xi + \|\, w \,\|^2 / 2 \,:\, \xi - Xw + bu \geq -y - \varepsilon u \,,\, \xi + Xw - bu \geq y - \varepsilon u \,,\, \xi \geq 0 \,\}$
and therefore its dual is

$$
\begin{aligned}
\max\ & \alpha^+(-y - \varepsilon u) + \alpha^-(y - \varepsilon u) - \|\, v \,\|^2 / 2 \\
& \alpha^+ + \alpha^- + s = Cu \\
& -\alpha^+ X + \alpha^- X - v = 0 \\
& \alpha^+ u - \alpha^- u = 0 \\
& \alpha^- \geq 0 \,,\, \alpha^+ \geq 0 \,,\, s \geq 0
\end{aligned}
$$

with α^+ the multipliers of the first set of constraints, α^- those of the second, and s those of $\xi \geq 0$. Again, the slack variable s can be eliminated by making the constraint a $\leq$ one. Then, the problem can be written in term of $\alpha = \alpha^+ - \alpha^-$ and $|\,\alpha\,| = \alpha^+ + \alpha^-$ (the latter being the element-wise absolute value vector), since in each optimal solution at least one between α_i^+ and α_i^- is 0 for each i. Indeed, if one had, say, $\alpha_i^+ > \alpha_i^- > 0$, doing $\alpha_i^+ \leftarrow \alpha_i^+ - \alpha_i^-$ and $\alpha_i^- \leftarrow 0$ the value of all terms $\alpha_i^+ - \alpha_i^-$ does not change while the value of all terms $\alpha_i^+ + \alpha_i^-$ decreases, hence the new solution is feasible (if the original one was) and it has a smaller objective value; thus the original solution could not be optimal. All in all, the dual can be rewritten $\max\{\, \alpha y - \varepsilon u|\,\theta\,| - \alpha^T(X^T X)\alpha \,/\, 2 \,:\, -Cu \leq \alpha \leq Cu \,,\, \alpha u = 0 \,\}$, which is not a QP but can easily be transformed into one by rewriting $\min\{\,|\,x\,|\,\}$ as $\min\{\, v \,:\, v \geq x \,,\, v \geq -x \,\}$. This requires one new variable for each α_i, and therefore yields a QP with as many variables as the original form (sans the s): however, in this form "half of the variables do not appear in the quadratic term", which is in general convenient [**back**]

- As previously seen in the Proximal Bundle case, (KKT-G) on the variables w gives $[\,\nabla\|\cdot\|^2\,/\,2\,](\,w^*\,) - \alpha^* YW = 0$ for SVM and $[\,\nabla\|\cdot\|^2\,/\,2\,](\,w^*\,) - (\,\alpha^+ - \alpha^-\,)W = 0$ for SVR (recall that constraints need be written as $\leq$, which explains the change in sign to YW and W). Computing b^* requires using (KKT-CS): for any i s.t. $\alpha^*_i = 0$ the corresponding constraint $y^i(wx^i - b) \geq 1 - \xi_i$ in (SVM-P) must be satisfied as equality, and if also $\alpha^*_i < C$ then $\xi^*_i = 0$ (recall that s_i is the dual variable of $\alpha_i \leq C$), which gives $y^i(w^*x^i - b^*) = 1$ that allows to compute b^* once w^* is obtained out of α^* (if $0 < \alpha^*_i < C$ happens for multiple i it may be a good idea numerically to compute b^* multiple times and take the average). Alternatively, if the solver provides (as it should) dual variables, b is just the dual variable of the $\alpha y = 0$ constraint. Similarly for (SVR-P), $w^*x^i - b^* - y_i - \varepsilon = 0$ whenever $C > \alpha^*_i > 0$ ($\equiv C > \alpha^{+,*}_i > 0$ and $\alpha^{-,*}_i = 0$) and $-w^*x^i + b^* + y_i - \varepsilon = 0$ whenever $-C < \alpha^*_i < 0$ ($\equiv C > \alpha^{-,*}_i > 0$ and $\alpha^{+,*}_i = 0$); or fetch the dual variable of the $\alpha u = 0$ constraint from the solver. This discussion justifies the "support vector" moniker. Starting from SVR, the points x^i s.t. $0 < \alpha^*_i < C$ are those that are correctly classified ($\xi^*_i = 0$) and that "lie on the boundary" of the two parallel classifying hyperplanes, i.e., $w^*x^i - b^* = 1$ or $w^*x^i - b^* = -1$. These are called "supporting vectors" of the hyperplane, and surely at least one exists (at least one point of one class will be correctly classified, and there is no point in having them all strictly in the interior of the classification zone). Eliminating all other points would not change the optimal dual solution α^*, and therefore nor w^* and b^*. Thus, like in the Proximal Bundle case, the dual optimal solution provides information about which points are "important" for the current classification (depending on the current choice of C). A similar description holds for SVR: the support hyperplanes are those on the border of the "insensitivity zone" $[\, y^i - \varepsilon \, , \, y^i + \varepsilon \,]$, picture two lines parallel to the graph of the function to be interpolated, one lifted above by ε and one below by the same amount. Since they are "on the border" they are correctly interpolated ($\xi_i^* = 0$), and are the ones which characterise the predicted function in the sense that even if all the other ones are removed from the fitting problem, the function remains the same [**back**]

- Assume we want to classify / interpolate using a cubic polynomial: we can map $x = [\, x_i \,]_{i=1,\dots,h}$ onto the vector having all possible $h(h-1)(h-2) / 6$ triples $x_p x_q x_m$ plus all possible $h(h-1) / 2$ pairs $x_p x_q$ plus all individual entries x_p; thus, the corresponding w would have $O(\, h^3 \,)$ entries. In general, a polynomial of degree k would entail $O(\, h^k \,)$ entries [**back**]

- Each term $\kappa(\, x \,,\, x^i \,) = e^{-\|\, x - x^i \,\|^2 \,/\, (2\sigma^2)}$ is 1 for $x = x^i$, but it will vanish quickly (the more so the more σ is small) as x drifts away from x^i. Thus, any function $f(\, x \,)$ could in principle be replicated with arbitrarily high accuracy by, roughly speaking, having "uncountably ∞-ly many" terms $\kappa(\, x \,,\, x^i \,)$ in the sum, one for each $x^i \in \mathbb{R}$, with $\alpha^*_i = f(\, x^i \,) + b^*$, and an "infinitely small σ". Note that the constraint $\sum_{i \in I} \alpha^*_i = 0$ is satisfied by taking $b^* = -\int f(\, x \,) dx$, provided of course that the integral is finite, which is guaranteed to hold if $f \in C^0$ and restricted to a finite interval $[\, x_- \,,\, x_+ \,]$. Thus, over any such finite interval, an appropriately large (but finite) number of support "vectors" $x^i \in \mathbb{R}$ and an appropriately small (but finite) σ should reasonably be able to reproduce any continuous function f. Of course this comes at the cost of a "very large data set" and it is very likely to result in "overfitting", i.e., it is not a good solution in terms of the bias/variance dilemma. Furthermore, this does not imply that SVR with Gaussian kernel is a universal approximator in the strong sense envisioned by ML, unlike, e.g., Neural Networks [13] [**back**]