$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = y_3$$

$$A \in \mathbb{R}^{4\times3} \qquad y_i = \sum_{i=1}^{4} A_{ij}x_j$$

$$y = \underbrace{\begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \\ A_{41} \end{bmatrix}}_{v_1} x_1 + \underbrace{\begin{bmatrix} A_{12} \\ A_{22} \\ A_{32} \\ A_{42} \end{bmatrix}}_{v_2} x_2 + \underbrace{\begin{bmatrix} A_{13} \\ A_{23} \\ A_{33} \\ A_{43} \end{bmatrix}}_{v_3} x_3$$

coordinates x_1, x_2, x_3

Linear system: given A, y, find coordinates x

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad y = \begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$

Many solutions:

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot 2 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot 4 + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot 0 = \begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix} \qquad A \cdot \underbrace{\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}}_{x} = y$$

$$v_1 \cdot 0 + v_2 \cdot 0 + v_3 \cdot 4 = y$$

1, 2, 2, ...

$y = \begin{bmatrix} 4 \\ 4 \\ 1 \\ 2 \end{bmatrix} \rightsquigarrow$ no solutions

Im A (Image) : set of vectors y reachable

Ker A (Kernel) : set of x s.t. $Ax = 0$

Exactly one solution if A is <u>invertible</u>:

A needs to be <u>square</u>, and there must exist another matrix A^{-1} s.t.

$$AA^{-1} = A^{-1}A = I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

⚠ On a computer, $y = A^{-1}x$ >> y = inv(A) * ~~x~~

Matlab: x = A \ y

Python: scipy.linalg.linsolve(A, y)

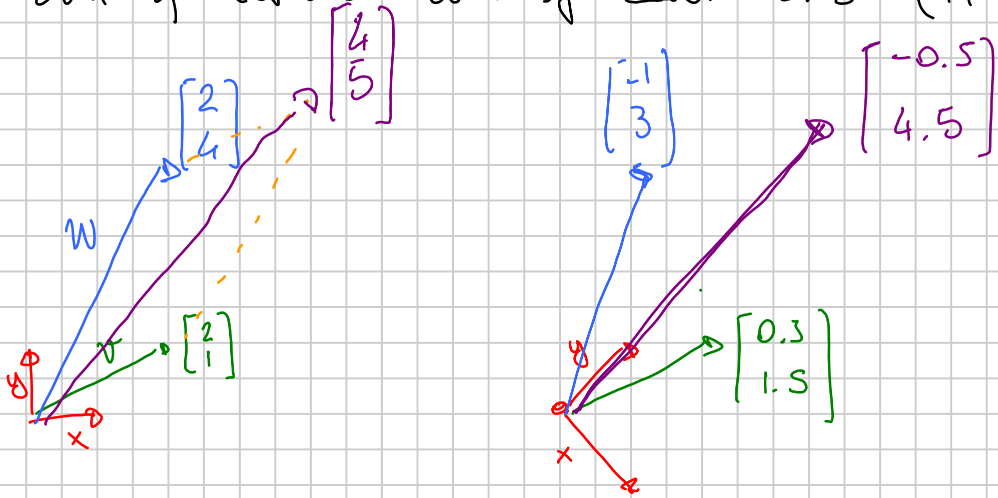
<u>Basis</u>: a tuple of vectors $v_1, \dots v_n \in \mathbb{R}^n$ s.t. every vector y can be written as linear combinations of them

$\forall y \ \exists c_1, \dots c_n \in \mathbb{R}$ s.t. $y = v_1 \cdot c_1 + \dots + v_n c_n$

Unique if $v_1 \dots v_n$ is a basis

$$\begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix} \circ + \dots + \begin{bmatrix} \\ \\ \end{bmatrix} \square$$

sum of vectors = sum of coordinates (in every basis!)



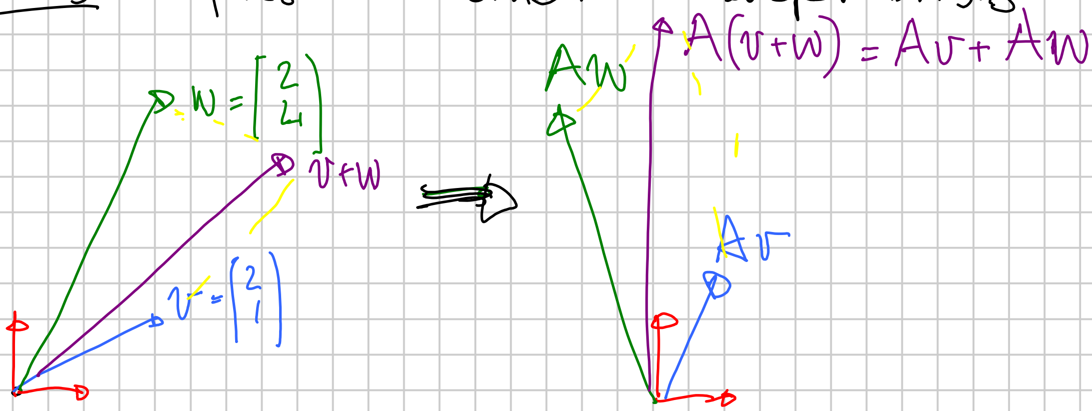
Simplest basis: canonical basis

e.g. in dimension $n=4$:

$$e_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad e_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad e_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad e_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix} = e_1 y_1 + e_2 y_2 + e_3 y_3 + e_4 y_4$$

<u>Matrices</u> represent linear transformations



e.g. rotation, scaling (in one or more direction), reflection, flattening/projection, ...

$I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$ identity matrix = identity function

AB product of matrices = composition of functions

$$A(Bv) = (AB)v$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ \vdots & & \\ A_{41} & A_{42} & A_{43} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ \vdots & \\ C_{41} & C_{412} \end{bmatrix}$$

$A \in \mathbb{R}^{4\times \boxed{3}}$ $\quad B \in \mathbb{R}^{\boxed{3}\times 2}$ $\longrightarrow$ $C \in \mathbb{R}^{4\times 2}$

$$C_{ik} = \sum_{j=1}^{\text{inner dim}} A_{ij} B_{jk}$$

$$C_{12} = \boxed{A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32}}$$ $n=3$

Consistent with the def. of matrix-vector product, if vector $\equiv$ matrix with 1 column

If $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, to compute AB you need mp entries, to compute each of which you need $2n-1$ operations

$\Rightarrow \quad mp(2n-1) = O(mnp)$ operations

flops (floating-point ops)

Usual algebra properties work: $A(B+C) = AB + AC$

$$(AB)C = A(BC)$$

$A, B \in \mathbb{R}^{n \times n}$ $\quad v \in \mathbb{R}^{n \times 1}$

$$\underbrace{\underbrace{(AB)}_{O(n^3)} v}_{O(n^2)} \qquad \Big| \qquad \underbrace{A \underbrace{(Bv)}_{O(n^2)}}_{O(n^2)}$$

⚠ Matlab does not rearrange parentheses to help you

$$(A+B)^2 = (A+B)(A+B) = A^2 + B^2 + AB + BA$$

⚠ $AB \neq BA$

$4\times3 \; 3\times2 \quad 3\times2 \; 4\times3$

⚠ $AC = BC \; \not\Rightarrow \; A = B$ if $C \neq 0$

If C is invertible, $AC = BC \Rightarrow A(CC^{-1}) = B(CC^{-1})$

$\Rightarrow A \cdot I = B \cdot I$

$\Rightarrow A = B$

---

row and column vectors:

$$c = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad r = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

3×1 $\qquad$ 1×3

Ac ✓ $\qquad$ ⚠ Ar does not exist if $A \in \mathbb{R}^{n \times n}$

Usually, variables = column vectors:

$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\quad w = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ $\quad w^T = [4 \;\; 5 \;\; 6]$

Scaler product:

$$\langle v, w \rangle = v^T w = [1 \;\; 2 \;\; 3] \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$$

outer product

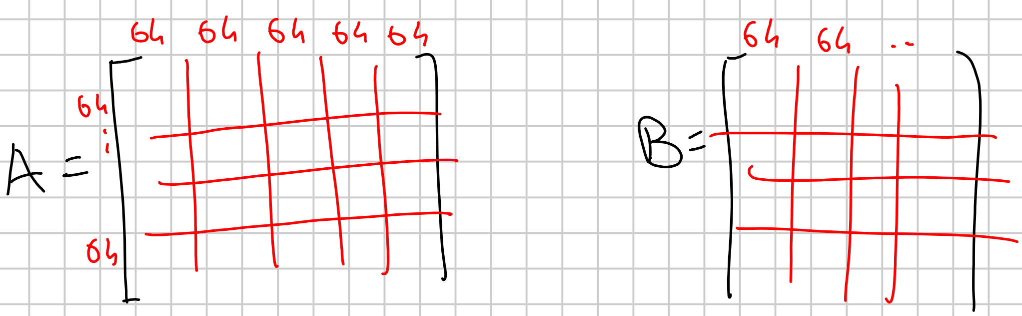
$$v\,w^T = \begin{bmatrix} \boxed{1} \\ 2 \\ 3 \end{bmatrix} [\boxed{4} \;\; 5 \;\; 6] = \begin{bmatrix} 1\cdot 4 & 1\cdot 5 & 1\cdot 6 \\ 2\cdot 4 & 2\cdot 5 & 2\cdot 6 \\ 3\cdot 4 & 3\cdot 5 & 3\cdot 6 \end{bmatrix}$$

Note: this is a rank-1 matrix

Def: A matrix has rank r iff its columns can be written as linear combinations of r distinct vectors (and I can't do it with $r-1$).

e.g. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\qquad A = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 2 \\ 4 & 2 & 2 \end{bmatrix}$ has rank 2

Theorem: column rank = row rank: if I replace "column" with "row" in the definition, I get the same number r



Numerical libraries often divide matrices into blocks for performance reasons when computing products (faster than vanilla for loops)

$$\begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \qquad \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

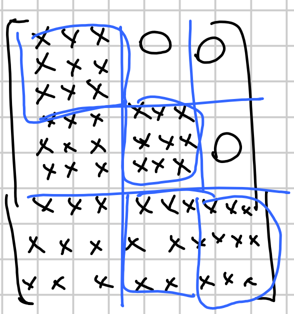
$$A_{11} x_1 = y_1 \qquad x_1 = \frac{y_1}{A_{11}}$$

$$A_{21} x_1 + A_{22} x_2 = y_2$$

$$\vdots$$

$O(n^2)$ operations ("forward substitution")

If you have a large matrix $3n \times 3n$ divided into 3 blocks, you can work in the same fashion:



$$\begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \qquad \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}$$

now each A_{ij} x_i y_j is a block

$$\begin{array}{cccc} & n & n & n \\ n & & & \\ n & & & \\ n & & & \end{array}\begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{array}{c} 1 \\ n \\ n \\ n \end{array}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{array}{c} 1 \\ n \\ n \\ n \end{array}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\boxed{A_{11}x_1 = y_1} \qquad x_1 = \frac{y_1}{A_{11}} \; !$$

$$A_{21}x_1 + A_{22}x_2 = y_2$$

$$\vdots$$

⚠ One cannot write $\frac{x}{A}$ if x is a vector and A is a matrix

$$A_{11}x_1 = y_1 \quad \Rightarrow \quad x_1 = A_{11}^{-1} y_1$$

$$A_{21}x_1 + A_{22}x_2 = y_2 \quad \Rightarrow \quad x_2 = A_{22}^{-1}\left(y_2 - A_{21}x_1\right)$$

$$\vdots \qquad\qquad \vdots$$