Norms: $v \mapsto \|v\|$ generalizes the absolute value

- $\|v\| \geq 0 \qquad \|v\| = 0 \iff v = 0$
- $\|v\alpha\| = \|v\| \cdot |\alpha|$
- $\|v + w\| \leq \|v\| + \|w\|$

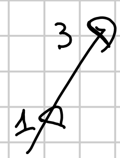
$$\|v\|_1 = \left\| \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\|_1 = |v_1| + |v_2| + \dots + |v_n|$$

$$\|v\|_\infty = \max_{i=1\dots n} |v_i|$$

$$\|v\|_2 = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} = (v^T v)^{\frac{1}{2}} = \left(\langle v, v\rangle\right)^{\frac{1}{2}}$$

Trivial but useful:

$v = u \cdot \alpha$



(with $u = \frac{v}{\|v\|} \quad \alpha = \|v\|$)

Def: a matrix U is called orthogonal if it is square and

- $U^T U = I$
- $U U^T = I$
- $U^{-1} = U^T$

(equivalent properties)

Property: $\|Ux\| = \|x\|$ (2-norm)

$$\|x\|^2 = x^T x \qquad \|Ux\|^2 = (Ux)^T(Ux) = x^T\underbrace{\boxed{U^T U}}_{= I}x = x^T x$$

$$x^T y = (Ux)^T(Uy)$$

Geometrically: rotations, symmetries.

Property:

- $(AB)^T = B^T A^T$ for all A, B
- $(AB)^{-1} = B^{-1}A^{-1}$ A, B square

"shoe-sock property"

Theorem: if U, V orthogonal, UV is orthogonal, too.

$$(UV)^{-1} = V^{-1}U^{-1} = V^T U^T = (UV)^T.$$

$U \in \mathbb{R}^{m\times m}$ orthogonal

$$U = \left[u_1 \middle| u_2 \middle| \cdots \middle| u_m\right]$$

Property: $u_i^T u_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ → the u_i's are orthonormal

$U^T U = I$

$$\begin{bmatrix} u_1^T \\ \hline u_2^T \\ \hline \vdots \\ \hline u_m^T \end{bmatrix} \left[u_1 \middle| u_2 \middle| \cdots \middle| u_m\right] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & 0 \\ 0 & \cdots & & 0 & 1 \end{bmatrix}$$

People call <u>orthogonal</u> a set of vectors that satisfies $u_i^T u_j = 0$ if $i \neq j$

Sometimes, we will work with matrices that are <u>rectangular</u> but with <u>orthonormal columns</u>

$$U_0 \in \mathbb{R}^{m \times n} \qquad m > n \qquad U_0 = \left[u_1 \middle| u_2 \middle| \cdots \middle| u_n \right]$$

$$u_i^T u_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Given any such matrix, I can always complete it to an orthonormal matrix

For all $U_0 \in \mathbb{R}^{m \times n}$ with this property, we can find $U_c \in \mathbb{R}^{m \times (m-n)}$ s.t. $U = [U_0 \ \ U_c] \in \mathbb{R}^{m \times m}$ is orthogonal. (square)

$$U_0^T U_0 \overset{?}{=} I \qquad\qquad U_0 U_0^T \overset{?}{=} I$$

True!   false!

$$I = U^T U = \begin{bmatrix} U_0^T \\ \hline U_c^T \end{bmatrix} \begin{bmatrix} U_0 & \vdots & U_c \end{bmatrix} = \left[\begin{array}{c|c} \overset{\overset{I}{\shortparallel}}{U_0^T U_0} & U_0^T U_c \\ \hline U_c^T U_0 & U_c^T U_c \end{array}\right]$$

$U_0 U_0^T \neq I$ always if $m > n$, because $U_0 U_0^T$ has rank n and I has rank m

Is it true that
- $\|U_0 x\| = \|x\|$ holds for all x?
- $\|U_0^T x\| = \|x\|$ for all x?

Eigenvalues, eigenvectors $\quad v \neq 0$ ⚠

Given $A \in \mathbb{R}^{m \times m}$, if $Av = v\lambda$ for $v \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$

then v is called eigenvector, λ eigenvalue

For many matrices, one can write

$$A = V \cdot \Lambda \cdot V^{-1} = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \\ & & & & \lambda_m \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_m^T \end{bmatrix}$$

with $A \cdot v_i = v_i \lambda_i \quad i = 1, 2, \dots m$

$[V, D] = \mathrm{eig}(A) \quad$ costs $O(m^3)$

Eigenvectors/values can be used to describe applying the same matrix to a vector many times

$$A \cdots (A(Ax)) = A^k x = (VD\boxed{V^{-1})(V}DV^{-1})\boxed{V} \cdots \boxed{(V}DV^{-1})\, x$$

$$= V \underbrace{D \cdot D \cdot \dots \cdot D}_{k \text{ times}} \cdot V^{-1} x = V \begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \ddots \\ & & & \lambda_m^k \end{bmatrix} V^{-1} x.$$

(pagerank is related)

Slight generalization: given a polynomial $p(t)$

$$p(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_d t^d \qquad (c_i \text{ real numbers})$$

$$p(A) = c_0 I + c_1 A + c_2 A^2 + \dots + c_d A^d$$

If $A = V \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix} V^{-1}$, then $p(A) = V \begin{bmatrix} p(\lambda_1) & & & \\ & p(\lambda_2) & & \\ & & \ddots & \\ & & & p(\lambda_m) \end{bmatrix} V^{-1}$.

Eigenvalues always well-defined.

Eigenvectors are non-unique: $Av = v\lambda$

$\Rightarrow v\alpha$ is also an eigenvector for each $\alpha \neq 0$

If repeated eigenvectors: $Av_1 = v_1\lambda_1 \quad Av_2 = v_2\lambda_2$

$\lambda_1 = \lambda_2$, then $v_1\alpha + v_2\beta$ is also a λ-eigenvector.

Extreme case: $I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = V \cdot I \cdot V^{-1}$

Also, eigenvectors can fail to exist in enough number to form a basis

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \lambda$

$v = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$

Spectral theorem: if A is a symmetric matrix (i.e., $A = A^T$), then we can always find U,

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_m \end{bmatrix} \quad \text{s.t.} \quad A = U \Lambda U^{-1}$$

Moreover, the λ_i are always real, and U can be taken <u>orthogonal</u>.

<u>Quadratic forms</u>: given a symmetric matrix $Q = Q^T$ $Q \in \mathbb{R}^m$, let us consider the function

$$x \longmapsto \underset{1\times m}{x^T}\,\underset{m\times m}{Q}\,\underset{m\times 1}{x}$$

$$\mathbb{R}^m \longmapsto \mathbb{R}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad Q = \begin{bmatrix} Q_{11} & \cdots & Q_{13} \\ \vdots & \ddots & \vdots \\ \cdots & & Q_{33} \end{bmatrix}$$

$$x^T Q x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ \vdots & & \\ Q_{31} & \cdots & Q_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$$Q_{11} x_1^2 + \boxed{Q_{12} x_1 x_2} + Q_{13} x_1 x_3 + \boxed{Q_{21} x_2 x_1} + Q_{22} x_2^2 + \cdots$$

One can always write an expression that is quadratic in the entries of x in this form.

E.g.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad x_1^2 + 2x_2^2 + 3x_1 x_2$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \begin{bmatrix} 1 & 3/2 \\ 3/2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Theorem:

Let Q be a symmetric matrix, with minimum eigenvalue λ_{min}, maximum λ_{max}.
Then, for all $x \in \mathbb{R}^m$

$$\lambda_{min} \|x\|^2 \leq x^T Q x \leq \lambda_{max} \|x\|^2$$

equivalently: $\lambda_{min} \leq u^T Q u \leq \lambda_{max}$ for any u with $\|u\|=1$

Proof: Simpler case: $Q = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_m \end{bmatrix}$

$$x^T Q x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}^T \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}^T \cdot \begin{bmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \\ \lambda_m x_m \end{bmatrix} = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_m x_m^2$$

$$\geq \lambda_{min} (x_1^2 + x_2^2 + \dots + x_m^2) = \lambda_{min} \|x\|^2$$

$$\leq \lambda_{max} (x_1^2 + \dots + x_m^2 = \lambda_{max} \|x\|^2$$

General case: Write $Q = U \cdot \Lambda \cdot U^T$

Define $c = U^T x \iff x = Uc$

$$x = \begin{bmatrix} u_1 | u_2 | \dots u_m \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = u_1 c_1 + u_2 c_2 + \dots + u_m c_m$$

c_m are coordinates of x in the basis of the eigenvectors of Q

$$x^T Q x = (Uc)^T U \Lambda U^T (Uc) = c^T (U^T U) \Lambda (U^T U) c$$

$$= c^T \Lambda c = [c_1 \dots c_m] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{cases} \leq \lambda_{max} \|c\|^2 \\ \geq \lambda_{min} \|c\|^2 \end{cases}$$

$x = Uc \Rightarrow \|x\| = \|c\|$ because U is orthogonal

If all the eigenvalues λ_i are ≥ 0, then

$$x^T Q x \geq 0$$

If this happens, we say Q is <u>positive semidefinite</u>

$Q \succcurlyeq 0$

If all eigenvalues are $\lambda_i > 0$, then

$$x^T Q x \geq \lambda_{min} \|x\|^2 > 0 \qquad \text{if } x \neq 0$$

If this happens, we say Q is <u>positive definite</u>

$Q \succ 0$

<u>Theorem:</u> for every $A \in \mathbb{R}^{m \times n}$,

AA^T and $A^T A$ are square, symmetric, positive semidefinite.

<u>Proof:</u> let's do $A^T A$ (the other case is analogous)

- $\underset{n\times m}{A^T}\,\underset{m\times n}{A} \to n \times n$
- $(A^T A)^T = A^T (A^T)^T = A^T A$
- $x^T A^T A x = (Ax)^T Ax = \|Ax\|^2 \geq 0$