Es 1:

$$f(x) = x - 1 + \frac{1}{x+1} \qquad x \neq -1$$

a) Condizionamento

$$\varepsilon_{in} = \frac{x}{f(x)} f'(x) \, \varepsilon_x$$

$$f(x) = x - 1 + \frac{1}{x+1} = \frac{x^2}{x+1}$$

$$f'(x) = 1 - \frac{1}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2}$$

$$\Rightarrow \varepsilon_{in} = \frac{\cancel{x}}{\cancel{x^2}} \cancel{(x+1)} \frac{\cancel{x}(x+2)}{(x+1)^{\cancel{2}}} \varepsilon_x = \frac{(x+2)}{x+1} \varepsilon_x$$

$$|\varepsilon_{in}| = \left| \frac{x+2}{x+1} \right| |\varepsilon_x| \leq \frac{|x+2|}{|x+1|} u$$

per $x \to -1$ il problema risulta malcondizionato

per $x \to \pm\infty$ il problema è invece ben condizionato

b) Stabilità: errore algoritmico.

$$g_1(x) = x - 1 + \frac{1}{x+1} \qquad g_2(x) = \frac{x^2}{x+1}$$

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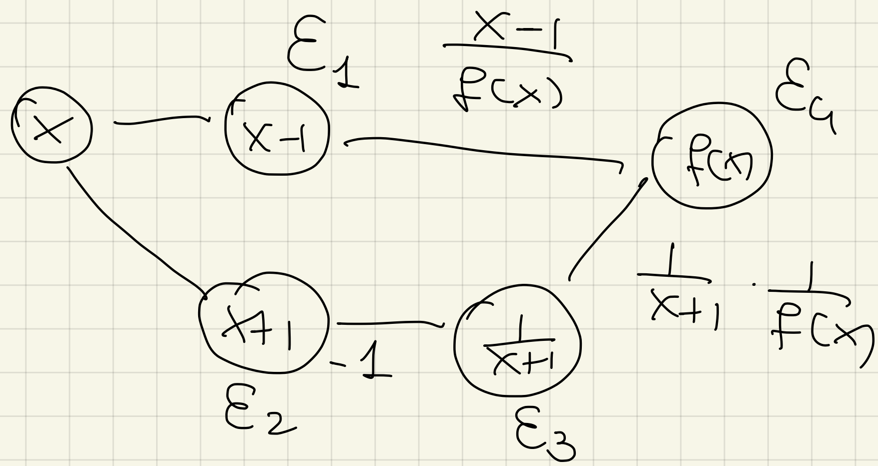



$$\varepsilon_{ALG}^{(1)} = \varepsilon_4 + \frac{x-1}{f(x)}\varepsilon_1 + \frac{1}{(x+1)}\cdot\frac{1}{f(x)}\left(\varepsilon_3 - \varepsilon_2\right)$$

$$\left|\varepsilon_{ALG}^{(1)}\right| = \left|\varepsilon_4 + \frac{x-1}{f(x)}\varepsilon_1 + \frac{1}{(x+1)}\cdot\frac{1}{f(x)}\left(\varepsilon_3 - \varepsilon_2\right)\right|$$

$$\leq |\varepsilon_4| + \frac{|x-1|}{|f(x)|}|\varepsilon_1| + \frac{1}{|x+1|}\cdot\frac{1}{|f(x)|}\left(|\varepsilon_3| + |\varepsilon_2|\right)$$

$$\leq u\left(1 + \frac{1}{|f(x)|}\left(|x-1| + \frac{2}{|x+1|}\right)\right)$$

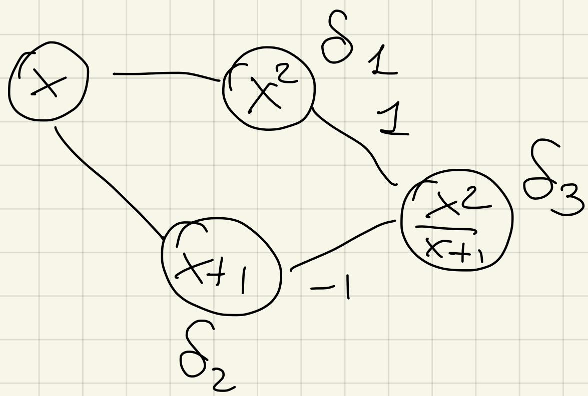
$$= u\left(1 + \frac{|x+1|}{x^2}\cdot\left(|x-1| + \frac{2}{|x+1|}\right)\right)$$

$$= u\left(1 + \frac{|x^2-1|}{x^2} + \frac{2}{x^2}\right)$$

Questo algoritmo è quindi instabile per $x \to 0$, mentre è stabile per $|x| \to \infty$

cancellazione numerica

② $g_2(x) = \dfrac{x^2}{x+1}$




$$\varepsilon^{(2)}_{ALG} = \delta_3 + \delta_1 - \delta_2$$

$|\varepsilon^{(2)}_{ALG}| \le 3u$ Questo algoritmo è sempre preferibile perché stabile per ogni valore di $x$.

Es 2:

$$A = I + \alpha e_n e_1^T + \alpha e_1 u^T \qquad \alpha \in \mathbb{R}$$

$$u = [0, 1 \dots 1]$$

$$e_n e_1^T = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [1\ 0 \cdots 0] = \begin{bmatrix} & & \\ & 0 & \\ 1 & & \end{bmatrix}$$

$$e_1 u^T = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [0, 1 \cdots 1] = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ & 0 & & \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} 1 & \alpha & \cdots & \alpha \\ & \ddots & & 0 \\ & 0 & \ddots & \\ \alpha & & & 1 \end{bmatrix}$$

a) Possiamo usare la pred. diagonale per colonne.
Se $|\alpha| < 1 \Rightarrow$ la matrice è a predominanza diagonale per colonne ed è quindi invertibile

col. 1: $\quad 1 > |\alpha|$

col 2 ⋯ n: $\quad |a_{jj}| = 1 > \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| = |a_{1j}| = |\alpha|$

b) Possiamo calcolare il det A ad esempio utilizzando la formula di Laplace, sviluppando rispetto alla prima colonna

$$\det A = 1 \cdot \det \underbrace{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}}_{= I_{n-1}} + (-1)^{n+1} \cdot \alpha \cdot \det \begin{bmatrix} \alpha & \cdots & \alpha \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$$= 1 + (-1)^{n+1} \alpha \cdot \alpha (-1)^n \det(I_{n-2}) =$$

$$= 1 - 1 \cdot \alpha^2 = 0 \iff \alpha^2 = 1 \iff |\alpha| = 1$$

c) $\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}| = 1 + |\alpha|$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| = \max\{(n-1)|\alpha|+1,\ 1,\ 1+|\alpha|\}$$

perché $n \geq 2$

$$\|A\|_\infty = (n-1)|\alpha|+1 \geq 1+|\alpha| = \|A\|_1$$

d) per $\alpha = 1$ $\quad A^TA = I + ee^T + e_1e_n^T + e_ne_1^T$

$$A^T = (I + e_1e_n^T + ue_1^T)$$

$$A^TA = (I + e_1e_n^T + ue_1^T)(I + e_ne_1^T + e_1u^T)$$

$$= I + e_1e_n^T + ue_1^T + e_ne_1^T + e_1\underbrace{e_n^Te_n}_{1}e_1^T +$$

$$+ e_1u^T + e_1\underbrace{e_n^Te_1}_{0}u + u\underbrace{e_1^Te_1}_{1}u^T$$

$$= I + e_1e_n^T + ue_1^T + e_ne_1^T + e_1e_1^T + e_1u^T + uu^T$$

$$= I + e_1e_n^T + (\overbrace{u+e_1}^{e})e_1^T + (\overbrace{e_1+u}^{e})u^T + e_ne_1^T$$

$$= I + e_1e_n^T + e(\underbrace{e_1^T+u^T}_{e^T}) + e_ne_1^T$$

$$= I + ee^T + e_1e_n^T + e_ne_1^T$$

$$= \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} + \begin{bmatrix} 1 & \cdots & 2 \\ \vdots & & \vdots \\ 2 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & \cdots & 1 & 2 \\ 1 & \ddots & & & 1 \\ \vdots & & \ddots & & \vdots \\ 2 & 1 & \cdots & 1 & 2 \end{bmatrix}$$

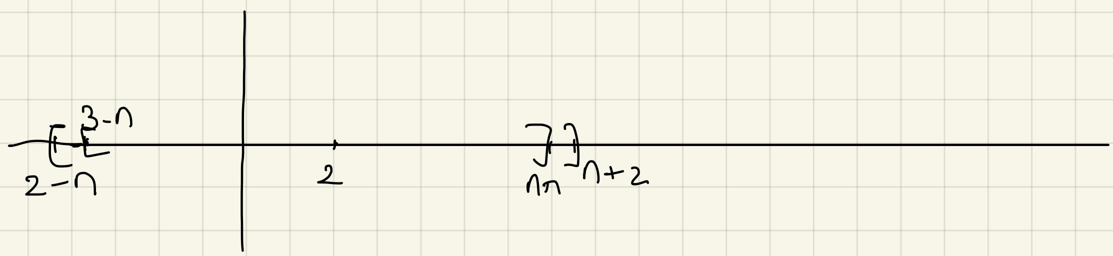
$$\|A\|_2 = \sqrt{\rho(A^TA)}$$

quindi per dimostrare che $\|A\|_2 \le \sqrt{n+2}$ occorre dimostrare che $\rho(A^TA) \le n+2$

Possiamo guardare con i cerchi di Gershgorin se sono possibili limitazioni al raggio spettrale.

$K_1 \equiv K_n$ centro $2$ e raggio $(n-2)+2 = n$

$K_2 \equiv K_3 \equiv \dots \equiv K_{n-1}$ centro $2$ e raggio $n-1$




$$\Rightarrow \rho(A^TA) \le \max\{n+2, |2-n|\} = n+2$$

$$\Rightarrow \|A\|_2 \le \sqrt{n+2}$$

e) Fatt LU

$$A = \begin{bmatrix} 1 & \alpha & \cdots & \alpha \\ & \ddots & & \\ & & \ddots & \\ \alpha & & & 1 \end{bmatrix}$$

$$L_{n-1} = \begin{bmatrix} I_{n-1} \end{bmatrix} \qquad U_{n-1} = \begin{bmatrix} 1 & \alpha & \cdots & \alpha \\ & \ddots & & \\ & & & 1 \end{bmatrix}$$

$$\begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \cancel{I}L_{n-1} \cdot y \qquad y = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$x^T U_{n-1} = (\alpha\, 0 \dots 0)$$

$$U_{n-1}^T \; x = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & & \\ \alpha & \ddots & \\ \vdots & & \\ \alpha & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

matrice elementare di gauss

$$\left(U_{n-1}^T\right)^{-1} = \begin{pmatrix} 1 & & \\ -\alpha & \ddots & \\ \vdots & & \\ -\alpha & & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_{n-1} \end{pmatrix} = \left(U_{n-1}^T\right)^{-1} \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & & \\ -\alpha & \ddots & \\ \vdots & & \\ -\alpha & & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ -\alpha^2 \\ \vdots \\ -\alpha^2 \end{pmatrix}$$

$$x^T y + \beta = 1 \qquad \beta = 1 - x^T y = 1 - (\alpha \; -\alpha^2 \dots -\alpha^2) \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= 1 - \alpha^2$$

$$L = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ \alpha & -\alpha^2 \dots & -\alpha^2 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & \alpha & \dots & \alpha & \alpha \\ & \ddots & & & \\ & & & 1 & \\ & & & & 1-\alpha^2 \end{pmatrix}$$

(f)

$$\begin{cases} L y = b \\ U x = y \end{cases} \qquad \begin{aligned} & y_{1:n-1} = b_{1:n-1} \\ & \alpha y_1 - \sum_{i=2}^{n-1} \alpha^2 y_i + y_n = 0 \end{aligned}$$

$$y_n = \sum_{i=2}^{n-1} \alpha^2 y_i - \alpha y_1$$

$$x_n = \frac{y_n}{1-\alpha^2}$$

$x_k = y_k$ $\quad k = n-1 : -1 : 2$

$$x_1 + \alpha x_2 + \dots + \alpha x_n = y_1$$

$$x_1 = y_1 - \sum_{i=2}^{n} \alpha x_i$$

```
function x=syssolve (b, alfa)
    n= lenght(b);
    y= b;
    y(n)= -alfa * y(1);
    for i= 2:n-1
        y(n)= y(n)+ alfa^2 * y(i)
    end
    x= y;
    x(n)= y(n)/ (1-alfa^2);
    for i=2:n
        x(i)= x(1)- alfa * x(i);
    end
end
```