Correzione del compitino del 7/3/2025

Es 1:

$$f(x) = \frac{x-a}{x^3}(x^2+ax+a^2) = 1 - \left(\frac{a}{x}\right)^3$$

a) Condizionamento

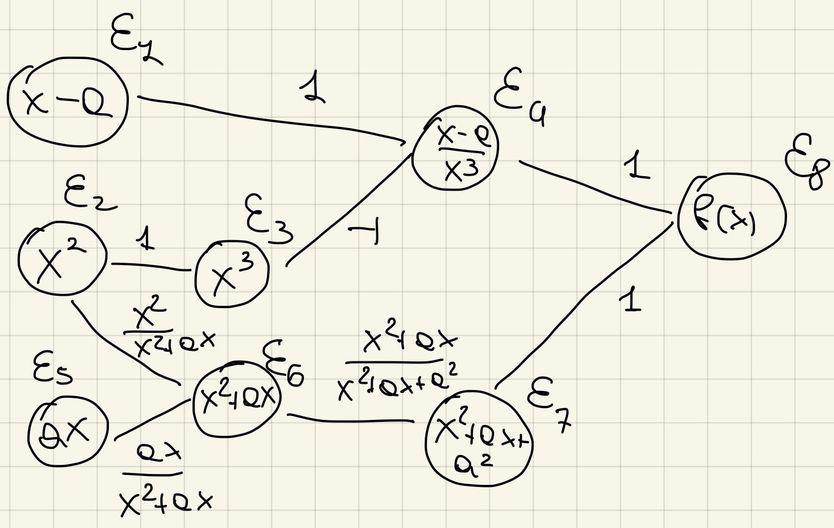
$$\varepsilon_{in} = \frac{x}{f(x)} f'(x)\, \varepsilon_x$$

$$= \frac{\cancel{x}}{x^3-a^3} \cdot \cancel{x^3} \cdot \frac{3a^3}{\cancel{x^4}} \varepsilon_x = \frac{3a^3}{x^3-a^3} \varepsilon_x$$

$$x^3 - a^3 = (x-a)(x^2+ax+a^2) = 0 \iff x = a$$

Quindi il problema risulta malcondizionato solo se $x \to a$

b) Alg 1



$$\varepsilon_{ALG}^{(1)} = \varepsilon_8 + 1\left(\varepsilon_4 + \varepsilon_1 - 1(\varepsilon_3 + \varepsilon_2)\right) +$$

$$+ 1\left(\varepsilon_7 + \frac{x^2+ax}{x^2+ax+a^2}\left(\varepsilon_6 + \frac{x^2}{x^2+ax}\varepsilon_2 + \frac{ax}{x^2+ax}\varepsilon_5\right)\right)$$

$$= \varepsilon_8 + \varepsilon_4 + \varepsilon_1 - \varepsilon_3 - \varepsilon_2 + \varepsilon_7 + \frac{x(x+a)}{x^2+ax+a^2}\varepsilon_6 + \frac{x^2}{x^2+ax+a^2}\varepsilon_2 + \frac{ax}{x^2+ax+a^2}\varepsilon_5$$

$$= \varepsilon_8 + \varepsilon_4 + \varepsilon_1 - \varepsilon_3 + \varepsilon_7 + \frac{x(x+a)}{x^2+ax+a^2}\varepsilon_6 + \frac{\cancel{x^2}-\cancel{x^2}-ax-a^2}{x^2+ax+a^2}\varepsilon_2 + \frac{ax}{x^2+ax+a^2}\varepsilon_5$$

$$|\varepsilon_{alg}^{(1)}| \le u\left(1+1+1+1+1+\frac{|x(x+a)|}{|x^2+ax+a^2|}+\frac{|a^2+ax|}{|x^2+ax+a^2|}+\frac{|ax|}{|x^2+ax+a^2|}\right)$$

si nota che $x^2+ax+a^2>0 \quad \forall x$ infatti $\Delta = a^2-4a^2<0$

In particolare il minimo è per $x = -\frac{a}{2}$ e vale

$$\left(-\frac{a}{2}\right)^2 + a\left(-\frac{a}{2}\right) + a^2 = \frac{a^2}{4} - \frac{a^2}{2} + a^2 = \frac{a^2-2a^2+4a^2}{4} = \frac{3}{4}a^2$$

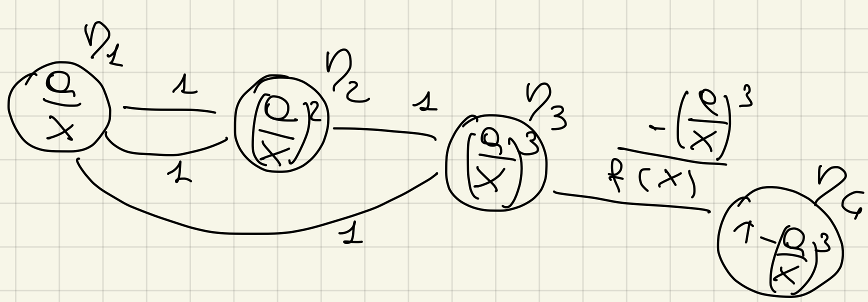
quindi $\frac{1}{|x^2+ax+a^2|} < \frac{4}{3}a^2$

Per $x \to \infty$ abbiamo che $\lim_{x\to\pm\infty}\left|\frac{x^2+ax}{x^2+ax+a^2}\right| = 1$

mentre $\lim_{x\to\pm\infty}\left|\frac{a^2+ax}{x^2+ax+a^2}\right| = \lim_{x\to\pm\infty}\left|\frac{ax}{x^2+ax+a^2}\right| = 0$

L'algoritmo risulta quindi non avere punti di instabilità

Alg 2:



$$\varepsilon_{ALG}^{(2)} = \eta_4 - \left(\frac{a}{x}\right)^3 \cdot \frac{x^3}{x^3-a^3}\left(\eta_3 + 1\left(\eta_2 + 2\eta_1\right) + \eta_1\right)$$

$$= \eta_4 - \frac{a^3}{x^3-a^3}\left(\eta_3 + \eta_2 + 3\eta_1\right)$$

$$|\varepsilon_{ACC}^{(2)}| \leq |\eta_4| + \frac{a^3}{|x^3-a^3|}\left(|\eta_3| + |\eta_2| + 3|\eta_1|\right)$$

$$\leq u\left(1 + \frac{5\,a^3}{|x^3-a^3|}\right)$$

questo algoritmo è quindi instabile per $x \to a$

c) $x = a - 10^{-3} \in \mathcal{F}$?

$a = 1 \Rightarrow x = 0.999 \in \mathcal{F}$

$a > 1 \Rightarrow$

$(a-1).999 = 10^1(.(a-1)999) \in \mathcal{F}$

Quindi per ogni a, $x \in \mathcal{F}$

$a = 1$ $fl\left(\frac{a}{x}\right) = 1.00100$

Calcoliamo l'algoritmo in aritmetica di $\mathcal{F}$

$fl\left(\frac{a}{x}\right)$

$a = 2$ $fl\left(\frac{a}{x}\right) = 1.00050$

$a = 3$ $fl\left(\frac{a}{x}\right) = 1.00033$

$a = 4$ $fl\left(\frac{a}{x}\right) = 1.00025$

$a = 5$ $fl\left(\frac{a}{x}\right) = 1.00020$

$a = 6$ $fl\left(\frac{a}{x}\right) = 1.00016$

$a = 7$ $fl\left(\frac{7}{x}\right) = 1.00014$

$a = 8$ $fl\left(\frac{8}{x}\right) = 1.00012$

$a = 9$ $fl\left(\frac{9}{x}\right) = 1.00011$

$fl\left(fl\left(\frac{a}{x}\right)^3\right) =$

$a = 1$	1.00300	$a = 8$	1.00036
$a = 2$	1.00150	$a = 9$	1.00033
$a = 3$	1.00099		
$a = 4$	1.00075		
$a = 5$	1.00060		
$a = 6$	1.00048		
$a = 7$	1.00042		

$fl\left(1 - fl\left(fl\left(\frac{a}{x}\right)^3\right)\right) =$

$a = 1$	$0.00300 = 0.3 \cdot 10^{-2}$
$a = 2$	$-0.00150 = -0.15 \ 10^{-2}$
$a = 3$	$-0.99 \ 10^{-3}$
$a = 4$	$-0.75 \cdot 10^{-3}$
$\vdots$	
$a = 9$	$-0.33 \cdot 10^{-3}$

ES 2.

a) La matrice M risulta

$$M = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} [0 1 \cdots 1 0] - \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} [0 1 \cdots 1 0] = \begin{bmatrix} 0 & 1 & \cdots & 1 & 0 \\ & & 0 & & \\ 0 & -1 & \cdots & -1 & 0 \end{bmatrix}$$

Vediamo che

$$M^2 = M \cdot M = (e_1 u^T - e_n u^T)(e_1 u^T - e_n u^T) = e_1 u^T e_1 u^T - e_n u^T e_1 u^T$$
$$- e_1 u^T e_n u^T + e_n u^T e_n u^T$$

$$u^T e_n = [0 1 \cdots 1 0] \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} = 0$$

$$u^T e_1 = 0$$

$$M^2 = 0$$

mentre $M^T M$

$$M^T M = (u e_1^T - u e_n^T)(e_1 u^T - e_n u^T) = u \underbrace{e_1^T e_1}_{1} u^T - u \underbrace{e_n^T e_1}_{0} u^T -$$
$$- u \underbrace{e_1^T e_n}_{0} u^T + u \underbrace{e_n^T e_n}_{1} u^T = 2 u u^T$$

$$M^TM = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & 2 & \cdots & 2 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & 2 & \cdots & 2 & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$ cioè la cornice di zero

b) $S = \{A : \; A = \alpha I + \beta M, \; \alpha \neq 0\}$

Gli autovalori di M sono tutti nulli, infatti:

$rk(M) = 1$ poiché $(n-2)$ righe sono nulle e le rimanenti due $M(1,:) = -M(n,:)$

$\Rightarrow$ M ha almeno $n-1$ autovalori nulli e l'autovalore restante da calcolare è uguale a traccia $(M) = 0$

Gli autovalori delle matrici in S hanno quindi autovalori $\alpha \cdot 1 + 0 \Rightarrow \lambda_A = \alpha$

quindi se $\alpha \neq 0 \Rightarrow A$ è invertibile

$$A_1 = \alpha_1 I + \beta_1 M$$

$$A_2 = \alpha_2 I + \beta_2 M$$

$$A_1 A_2 = (\alpha_1 I + \beta_1 M)(\alpha_2 I + \beta_2 M) = \alpha_1\alpha_2 I + \beta_1\alpha_2 M + \alpha_1\beta_2 M + \cancel{\beta_1\beta_2 M^2}$$

$$= \underbrace{\alpha_1\alpha_2}_{= \alpha} I + \underbrace{(\beta_1\alpha_2 + \alpha_1\beta_2)}_{= \beta} M \in S$$

Supponiamo che $A^{-1} \in S \Rightarrow \exists \; \delta, \gamma : \quad A^{-1} = \delta I + \gamma M$

Impongo che $AA^{-1} = I$ e determino δ, γ in funzione di α, β parametri di A

$$(\alpha I + \beta M)(\delta I + \gamma M) = \alpha\delta I + \beta\delta M + \gamma\alpha M + \cancel{\beta\gamma} M^2 = I$$

$$\Leftrightarrow \begin{cases} \alpha\delta = 1 \\ \beta\delta + \gamma\alpha = 0 \end{cases} \Rightarrow \quad \delta = \frac{1}{\alpha}$$

$$\frac{\beta}{\alpha} + \gamma\alpha = 0 \qquad \gamma\alpha = -\frac{\beta}{\alpha}$$

$$\gamma = -\frac{\beta}{\alpha^2}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{\alpha} I - \frac{\beta}{\alpha^2} M \quad \in S$$

$$A = \frac{a}{2} I + \frac{1}{n} M \qquad A^{-1} = \frac{2}{a} I - \frac{4}{na^2} \cdot M$$

c) $A = \frac{a}{2} I + \frac{2}{n} M$ è fatt LU

$$A = \begin{bmatrix} \frac{a}{2} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 \\ & \ddots & & & \\ 0 & -\frac{1}{n} & \cdots & -\frac{1}{n} & \frac{a}{2} \end{bmatrix} \quad \text{è fatt. } LU$$

perché

$$A(1:k, 1:k) = \begin{bmatrix} \frac{a}{2} & \frac{1}{n} & \cdots & \frac{1}{n} \\ & \ddots & & \\ & & & \frac{a}{2} \end{bmatrix}$$

$$\det A(1:k, 1:k) = \left(\frac{a}{2}\right)^k \neq 0 \qquad \text{perché } a \neq 0$$

Quindi la fatt. esiste ed è unica

$$\begin{bmatrix} \frac{a}{2} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 \\ & \ddots & & & \\ 0 & -\frac{1}{n} & \cdots & -\frac{1}{n} & \frac{a}{2} \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ \hline & x^T & & 1 \end{array}\right] \left[\begin{array}{ccc|c} \frac{a}{2} & \frac{1}{n} \cdots & \frac{1}{n} & \\ & \ddots & & y \\ & & \frac{a}{2} & \\ \hline & 0^T & & \beta \end{array}\right]$$

$$I_{n-1} \cdot y = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad y = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$x^T \begin{bmatrix} \frac{a}{2} & \frac{1}{n} & \cdots & \frac{1}{n} \\ & \ddots & & \\ & & & \frac{a}{2} \end{bmatrix} = \left(0, \frac{2}{n}, \cdots, \frac{2}{n}\right)$$

$$\begin{bmatrix} \frac{a}{2} & & \\ \frac{1}{n} & \ddots & \\ \vdots & & \\ \frac{1}{n} & & \frac{a}{2} \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{n} \\ \vdots \\ -\frac{1}{n} \end{pmatrix}$$

$$\frac{a}{2} x_1 = 0 \qquad x_1 = 0$$

$$\frac{1}{n} x_1 + \frac{a}{2} x_2 = -\frac{1}{n} \Rightarrow x_2 = \frac{-2}{na}$$

$$\vdots$$

$$x_{n-1} = -\frac{2}{na}$$

$$x = \left(0, -\frac{2}{na} \cdots -\frac{2}{na}\right)^T$$

$$x^T y + \beta = \frac{a}{2}$$

poiché $y = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow x^T y = 0$

$$\Rightarrow \beta = \frac{a}{2}$$

$$L = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ 0 & & & 1 \\ 0 \;\; -\frac{2}{na} & \cdots & -\frac{2}{na} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} \frac{a}{2} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 \\ & \ddots & & & \\ & & & & \frac{a}{2} \end{bmatrix}$$

$$\det U = \left(\frac{a}{2}\right)^n$$

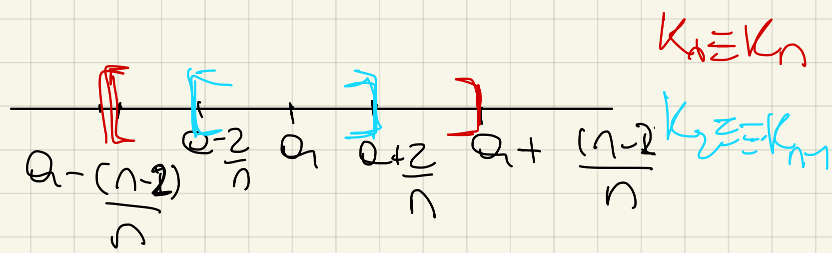
d) sia $B = A + A^T = \frac{a}{2} I + \frac{1}{n} M + \frac{a}{2} I + \frac{1}{n} M^T$

la matrice è simmetrica per cui $\|B\|_2 = \rho(B)$ e $\|B^{-1}\| = \frac{1}{|\lambda_{min}|}$

$B = aI + \frac{1}{n}(M + M^T)$ gli autov. sono reali

$$\begin{bmatrix} a & \frac{1}{n} & - - & \frac{1}{n} & 0 \\ \frac{1}{n} & \ddots & & & -\frac{1}{n} \\ & & 0 & & \vdots \\ \frac{1}{n} & 0 & & \ddots & -\frac{1}{n} \\ 0 & -\frac{1}{n} & \cdots & -\frac{1}{n} & a \end{bmatrix}$$

Cerchi di Gershgorin:



$$\boxed{0 < a - \frac{(n-2)}{n} \le \lambda_i \le a + \frac{(n-2)}{n}}$$

$$\|B\|_2 = \rho(B) \le a + \frac{(n-2)}{n}$$

$$\|B^{-1}\|_2 = \frac{1}{|\lambda_{min}|} < \frac{1}{\frac{na - n + 2}{n}} = \frac{n}{na - n + 2}$$

$$\text{cond}_2(B) \le \frac{na + n - 2}{\cancel{n}} \cdot \frac{\cancel{n}}{na - n + 2} = \frac{na + n - 2}{na - n + 2}$$

Per $a = 1$ $\quad$ cond $B \le n - 1$

per $a \geq 2$ $\quad$ cond $B \le 3$.