1 Identity-plus-rank-1 matrices

Let $u, v \in \mathbb{C}^n$ be vectors. Recall that the product u^*v produces a scalar (i.e., a number, $u^*v \in \mathbb{C}$), while vu^* produces a $n \times n$ matrix whose columns are all multiples of v (i.e., a rank-1 matrix).

A square matrix of the form $M = I + vu^*$ is called *identity plus rank 1*, or *rank-1 perturbation of the identity matrix*, or sometimes also *elementary matrix*.

Why are these matrices useful? First of all, it is easy to compute the product Mx for any vector x in time $\Theta(n)$: since $Mx = (I + vu^*)x = x + v(u^*x)$, and u^*x is a scalar, a simple algorithm is the following (in Matlab).

```
function y = product(u, v, x)
% computes the product Mx, where M = I + vu^*.
k = u' * x; %scalar product of two vectors
y = x + v * k; % product between a scalar and a vector
               % then sum of two vectors
```

Normally, the product between a $n \times n$ matrix and a vector costs $\Theta(n^2)$.

Theorem 1. *Let $u, v \in \mathbb{C}^n$, and set $M = I + vu^* \in \mathbb{C}^{n\times n}$. Then, M is invertible if and only if $u^*v \neq -1$, and in this case its inverse is $M^{-1} = I - \frac{1}{1+u^*v}vu^*$*

The fact to remember here is that the inverse of an identity-plus-rank-1 matrix is still a matrix of the same form, $M^{-1} = I + \alpha vu^*$, for some scalar α. If we need to remember the exact value of α, we can get the exact value of α by expanding the product

$$\begin{aligned} MM^{-1} &= (I + vu^*)(I + \alpha vu^*) \\ &= I + vu^* + \alpha vu^* + vu^*\alpha vu^* \\ &= I + vu^* + \alpha vu^* + \alpha v(u^*v)u^* \\ &= I + (1 + \alpha + \alpha u^*v)vu^*, \end{aligned}$$

where we have used the fact that α and u^*v are scalar and so we can move them in any position in the product. For this matrix to be equal to the identity, we need $1 + \alpha + \alpha u^*v$, which we can solve in α to get $\alpha = -\frac{1}{1+u^*v}$.

We can turn this argument into a formal proof with minor changes.

Proof of Theorem 1. Let us first consider the case $v^*u \neq -1$. We set $\alpha = -\frac{1}{1+v^*u}$ and $N = I + \alpha vu^*$. It holds that

$$MN = I + (1 + \alpha + \alpha u^*v)vu^*$$

following the same steps as above. With our choice of α we have $(1+\alpha+\alpha u^*v) = 0$, so the right-hand side reduces to I. This shows that $MN = I$, hence N is the inverse of M.

Since we formulated the theorem with an ‘if and only if’, we still need to show that if $u^*v = -1$ then M is not invertible. In this case we compute

$$Mv = (I + vu^*)v = v + v(u^*v) = v - v = 0,$$

so we can use the following fact from linear algebra: *for any square matrix M, if there exists a nonzero vector v such that $Mv = 0$ then M is not invertible.*

This last part assumes $v \neq 0$; but the case $v = 0$ is trivial since in this case $M = I$, and we can verify the claim of the theorem directly. □

For a generic $n \times n$ matrix, finding the inverse costs $\Theta(n^3)$, instead.