Exercises (Householder reflectors and QR factorization)

1. In Matlab, choose a suitable sequence of points in the plane $\left[\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right]$, $i = 1, 2, \ldots, n$, store them in a matrix $A \in \mathbb{R}^{2\times n}$, and plot them to create a recognizable picture; for instance

```
A = [0 1 1 0 0 0.5 0.7 0.7 0.8 0.8 1 1 0.6 0.6 0.4 0.4;
     1 1 0 0 1 1.5 1.3 1.4 1.4 1.2 1 0 0 0.3 0.3 0];
plot(A(1,:), A(2,:))
```

2. Now take a Householder reflector $H = I - \frac{2}{u^*u}uu^*$ (with a vector u of your choice), compute the points $H\left[\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right]$ and plot them in the same way. How is the picture transformed? Note that the orientation of the picture is reversed.

3. (Can you compute all the points $H\left[\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right]$ in one instruction without a `for` cycle?)

4. Now take any other matrix at your choice $M \in \mathbb{R}^{2\times 2}$ instead of H, and repeat the exercise. Typically M will not be orthogonal (check by computing `M'*M`). How is the picture transformed?

5. The matrix $G = \left[\begin{smallmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{smallmatrix}\right]$ represents a rotation in the 2×2 plane. Compute and plot the points $G\left[\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right]$. Is the orientation reversed this time?

6. Compute the eigenvalues of a Householder reflector $H = I - \frac{2}{u^*u}uu^*$ with Matlab's instruction `eig`. Do this both for $u \in \mathbb{R}^2$ and for larger dimensions.

7. Think about the geometrical picture of a Householder reflector: can you identify an eigenvector associated to the eigenvalue -1, i.e., a vector v such that $Hv = -v$? Can you identify some vectors z associated to the eigenvalue 1?

8. Choose a point of the sequence you created in the first exercise, for instance `A(1:end, 7)`. Construct a Householder reflector $\hat{H}$ that transforms this point into a multiple of e_1, and plot the points $\hat{H}\left[\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right]$ as you did before. Check graphically that the chosen point ends up on the x axis.

9. As you know, there are *two* real Householder reflectors that transform a point into a multiple of e_1. Repeat the previous exercise with the other reflector.

10. Think about the QR factorization of a 'short fat' matrix $M \in \mathbb{R}^{m\times n}$, $m < n$. How must the zero pattern of R be?

11. Let M be a 'short fat' matrix $M \in \mathbb{R}^{m\times n}$, and partition it as $\begin{bmatrix} M_1 & M_2 \end{bmatrix}$, where M_1 is square. What is the size of M_2, as a function of m and n? Show that a QR factorization $M = QR$ is given by $Q = \hat{Q}, R = \begin{bmatrix} \hat{R} & \hat{Q}^* M_2 \end{bmatrix}$, where $\hat{Q}, \hat{R}$ are the QR factors of M_1.