Exercises (Schur form)

1. (a) Let $T \in \mathbb{C}^{n\times n}$ be an upper triangular matrix, partitioned in blocks as
$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33}, \end{bmatrix}$$
where T_{11} and T_{33} are square, and T_{22} is 1×1 (i.e., the second block has width 1), and contains the value z. The matrix $T - zI$ is singular (since it is triangular with a 0 on the diagonal). Show that if T_{11} does not have the value z on its diagonal then the vector
$$\begin{bmatrix} -(T_{11} - zI)^{-1}T_{12} \\ 1 \\ 0 \end{bmatrix}$$
is in the kernel of $T - zI$.

   (b) Let $T \in \mathbb{C}^{n\times n}$ be an upper triangular matrix, such that its diagonal entries $t_{11}, t_{22}, \dots, t_{nn}$ are all different. Find an algorithm to compute its eigenvectors.

   (c) Find an algorithm to compute an eigenvector for each distinct eigenvalue of a matrix A given the factors Q, T of its Schur form $A = QTQ^{-1}$. Your algorithm should have complexity $O(n^3)$ (check this!).

2. (a) Let $A \in \mathbb{C}^{n\times n}$ be block triangular, i.e.,
$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}.$$
Given the factors of the Schur forms of $A_{11} = Q_{11}T_{11}Q_{11}^*$ and $A_{22} = Q_{22}T_{22}Q_{22}^*$, can you compute explicitly the Schur form of A?

   (b) Use the previous point to give another proof that the eigenvalues of A are given by the union of the eigenvalues of A_{11} and those of A_{22}.

3. A matrix $A \in \mathbb{C}^{n\times n}$ is called *normal* if $A^*A = AA^*$.

   (a) Show that A is normal if and only if the triangular factor of its Schur form T is normal.

   (b) Show that an upper triangular matrix T is normal if and only if it is diagonal. (Hint: compute the diagonal entries of TT^* and T^*T).

(c) Use the previous two points to show that A is normal if and only if in its Schur form the factor T is diagonal.