Limiti fondamentali

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$\frac{1-\cos x}{x^2} = \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}$$

$$= \frac{1-\cos^2 x}{x^2(1+\cos x)} = \frac{\sin^2 x}{x^2(1+\cos x)}$$

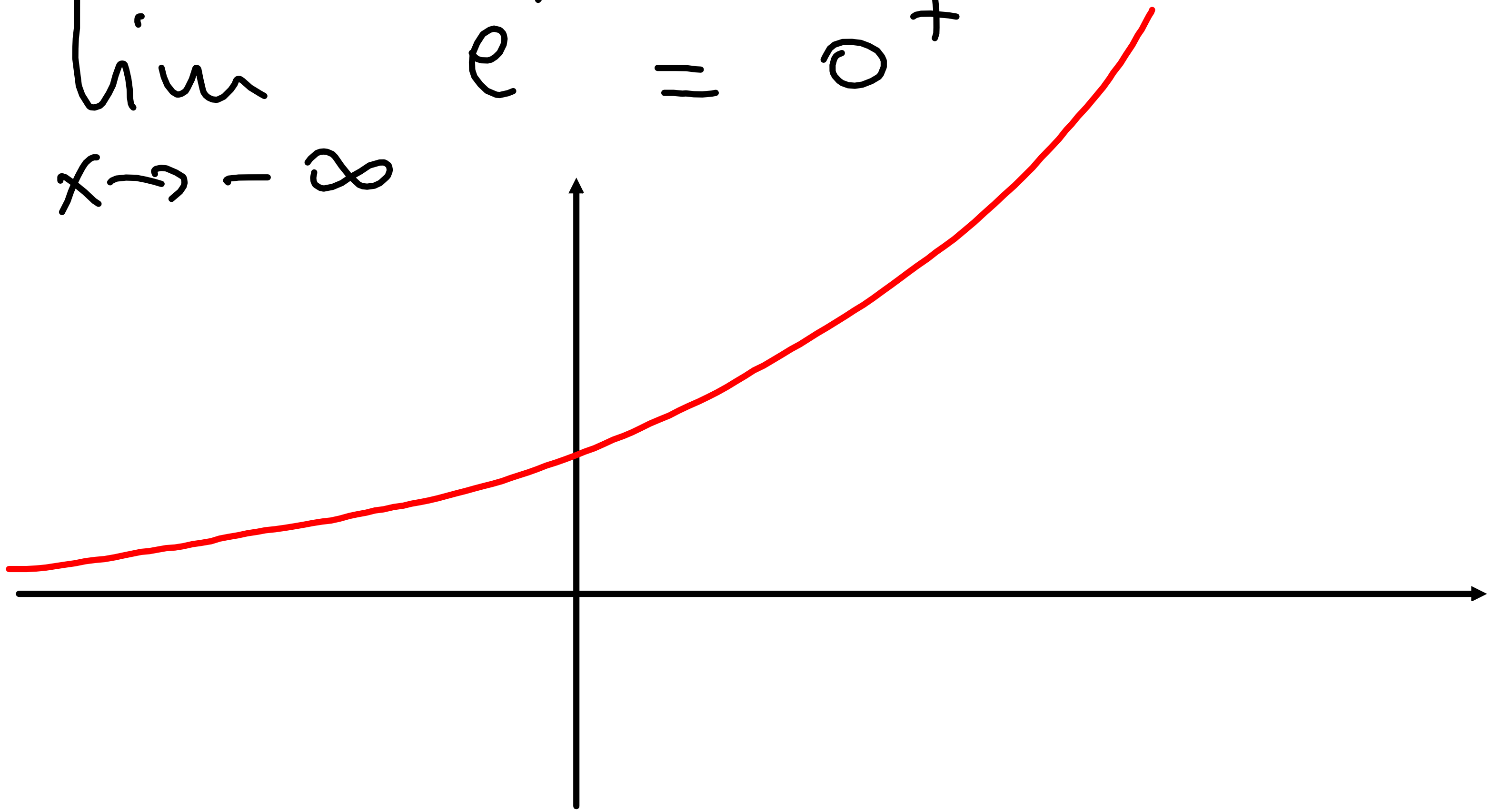
$$= \underbrace{\frac{\sin x}{x}}_{\downarrow\, 1} \cdot \underbrace{\frac{\sin x}{x}}_{\downarrow\, 1} \cdot \underbrace{\frac{1}{1+\cos x}}_{\downarrow\, \frac{1}{1+1}} \Rightarrow \frac{1}{2}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$$

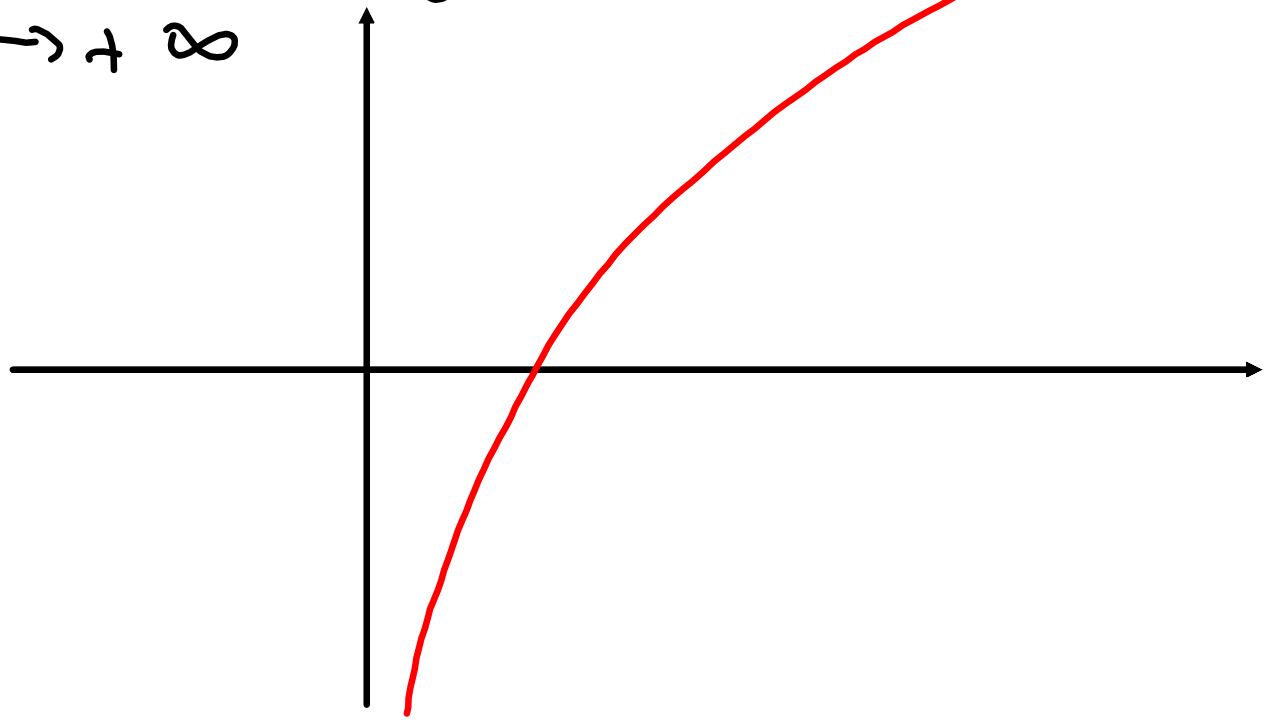
$$\lim_{x \to +\infty} e^x = +\infty$$

$$\lim_{x \to -\infty} e^x = 0^+$$



$$\lim_{x \to 0^+} \log x = -\infty$$

$$\lim_{x \to +\infty} \log x = +\infty$$



Limite della composizione

$A, B \subset \mathbb{R}$ $\quad f : A \to B$, $g : B \to \mathbb{R}$

$x_0 \in Acc(A)$. Se esiste

$\lim_{x \to x_0} f(x) = y_0$ e $y_0 \in Acc(B)$

allora

1) Se $y_0 \in B$ e g è continua in y_0 allora

$$\lim_{x \to x_0} (g \circ f)(x) = g(y_0)$$

2) Se $\lim_{y \to y_0} g(y) = \ell$ e $\exists$ $U \in \mathcal{I}(x_0)$ t.c. $f(x) \neq y_0$ $\forall x \in U \cap A \setminus \{x_0\}$ allora

$$\lim_{x \to x_0} (g \circ f)(x) = \ell .$$

Es: $\lim_{x \to -\infty} \operatorname{arctg}(x^2)$

$f: \mathbb{R} \longrightarrow \mathbb{R} \quad f(x) = x^2$

$g: \mathbb{R} \longrightarrow \mathbb{R} \quad g(y) = \operatorname{arctg} y$

$(g \circ f)(x) = g(f(x)) = g(x^2) =$

$= \operatorname{arctg}(x^2)$

$x_0 = -\infty$.

$$y_0 = \lim_{x \to x_0} f(x) = \lim_{x \to -\infty} x^2 = +\infty$$

siamo nel caso 2) perché

$+\infty \notin B = \mathbb{R}$

$$\ell = \lim_{y \to y_0} g(y) = \lim_{y \to +\infty} \operatorname{arctg}(y)$$

$$= \pi/2$$

devo verificare che $\exists\, V \in \mathcal{I}(x_0)$

t.c. $x \in A \cap V \setminus \{x_0\} \Rightarrow f(x) \neq y_0$

ma $y_0 = +\infty$ quindi la condizione

vale per un qualsiasi V

(il caso 2) è sempre verificato

se $y_0 = \pm\infty$).

$\Rightarrow$ per il teorema

$$\lim_{x \to x_0} (g \circ f)(x) = \ell \quad \text{cioè}$$

$$\lim_{x \to -\infty} \operatorname{arctg}(x^2) = \frac{\pi}{2}.$$

x_0

ℓ

È un teorema di cambiamento di variabile nel limite.

$$\lim_{x \to -\infty} \operatorname{arctg}(x^2)$$

pongo $y = x^2$ (cambio variabile)

la cambio da tutte le parti

Se $x \to -\infty$ allora

$$y = x^2 \longrightarrow +\infty$$

$$\lim_{x \to -\infty} \operatorname{arctg}(x^2) = \lim_{y \to +\infty} \operatorname{arctg}(y)$$

$$= \frac{\pi}{2} .$$

Limiti fondamentali

$a > 0$

$$\lim_{x \to \infty} a^x = \begin{cases} +\infty & \text{se } a > 1 \\ 1 & \text{se } a = 1 \\ 0^+ & \text{se } 0 < a < 1 \end{cases}$$

$$\lim_{x \to -\infty} a^x$$

metto $y = -x$

$$= \lim_{y \to +\infty} a^{-y}$$

$y = -x$ se $x \to +\infty$

$\Downarrow$ $\Rightarrow y \to -\infty$

$x = -y$

$$\lim_{x \to -\infty} a^x = \lim_{y \to +\infty} a^{-y} =$$

$$= \lim_{y \to +\infty} \frac{1}{a^y} = \begin{cases} 0^+ & \text{se } a > 1 \\ 1 & \text{se } a = 1 \\ +\infty & \text{se } 0 < a < 1 \end{cases}$$

$$\lim_{x \to \infty} x^{\alpha} = \begin{cases} \infty & \text{se } \alpha > 0 \\ 1 & \text{se } \alpha = 0 \\ 0^{+} & \text{se } \alpha < 0 \end{cases}$$

$\alpha \in \mathbb{R}$

$$\lim_{x\to\infty} \frac{a^x}{x^\alpha} = \begin{cases} +\infty & \text{se } a>1 \\ 0^+ & \text{se } 0<a<1 \end{cases}$$

$a, \alpha \in \mathbb{R}$

$a > 0$

$a = 1$?

$$\frac{a^x}{x^\alpha} = \frac{1}{x^\alpha}$$

Es: $a = \frac{1}{2}$ $\quad \alpha = -3$

$$\frac{a^x}{x^\alpha} = \frac{\left(\frac{1}{2}\right)^x}{x^{-3}} = \frac{x^3}{2^x} \to 0^+$$

Es:

$$\lim_{x\to\infty} \frac{\log x}{x} = \circledast$$

cambio di variabile

$$y = \log x \iff x = e^{y}$$

se $x \to \infty \Rightarrow y \to \infty$

$$\circledast = \lim_{y\to\infty} \frac{y}{e^{y}} = 0$$

Es: $\lim_{x\to+\infty} \frac{(\log x)^\beta}{x^\alpha}$ $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 0$.

cambiamento di variabile $\log x = y$. quindi $x = e^y$

se $x \to +\infty$ allora $y = \log x \to +\infty$. Il limite diventa

$$\lim_{y\to+\infty} \frac{y^\beta}{(e^y)^\alpha} = \lim_{y\to+\infty} \frac{y^\beta}{e^{y\alpha}} = \lim_{y\to+\infty} \frac{y^\beta}{(e^\alpha)^y} = 0$$

perché $e^\alpha > 1$ dato che $\alpha > 0$.

$$\lim_{x \to 0^+} (1+x)^{1/x} =$$

$$\log\left[(1+x)^{1/x}\right]$$

$$= \lim_{x \to 0^+} e$$

$$\frac{1}{x}\log(1+x)$$

$$= \lim_{x \to 0^+} e \qquad y = \frac{1}{x}\log(1+x)$$

$$\lim_{x \to 0^+} \frac{1}{x} \log(1+x) = 1$$

$$x \to 0^+ \implies y \to 1$$

$$\lim_{x \to 0^+} e^{\frac{1}{x} \log(1+x)} = \lim_{y \to 1} e^{y} = e^{1} = e$$

Nuovi casi di indeterminazione

$f(x) > 0$

$$\lim_{x \to x_0} f(x)^{g(x)} = ?$$

quando può dare indeterm.?

$$f(x)^{g(x)} = e^{\log\left[f(x)^{g(x)}\right]}$$

$$= e^{g(x)\log(f(x))}$$

quando $g(x)\cdot\log(f(x))$

è indeterminato?

1) $g \to 0$ $\qquad \log(f) \to +\infty$

$$\Downarrow$$

$f \to +\infty$

$f^{g} = (+\infty)^{0}$

2) $g \to 0$ $\qquad \log(f) \to -\infty$

$$\Downarrow$$

$f \to 0^{+}$

$f^{g} = (0^{+})^{0}$

3) $g \to \pm\infty$ $\quad \log f \to 0$

$$\Downarrow$$

$$f \to 1$$

$$f^{g} = (1)^{\pm\infty}$$

Es: $\lim_{x \to 0^+} x \log x = 0^+ \cdot (-\infty)$?

$y = \log x \iff x = e^y$

se $x \to 0^+ \implies y \to -\infty$

$$\lim_{y \to -\infty} e^y \cdot y = 0^+ (-\infty)$$

$$y = -z$$

$$y \to -\infty \implies z \to \infty$$

$$\lim_{y \to -\infty} e^{y} \cdot y = \lim_{z \to \infty} e^{-z}(-z) =$$

$$= \lim_{z \to \infty} -\frac{z}{e^{z}} = -(0^{+}) = 0^{-}$$

$$\lim_{x \to 0^+} x^\alpha \log x = ?$$ $\alpha > 0$

sostituisco $y = x^\alpha$

$\Longleftrightarrow \quad x = y^{1/\alpha}$

$x \to 0^+ \Rightarrow y \to 0^+$

$$\Rightarrow \lim_{x \to 0^+} x^\alpha \log x =$$

$$= \lim_{y \to 0^+} y \log\left(y^{1/\alpha}\right) =$$

$$= \frac{1}{\alpha} \lim_{y \to 0^+} y \log y = 0$$

Es:

$$\lim_{x\to 0^+} x^x =$$

$$= \lim_{x\to 0^+} e^{\log(x^x)} =$$

$$= \lim_{x\to 0^+} e^{x\log x} \xrightarrow{} 0^- = e^0 = 1$$

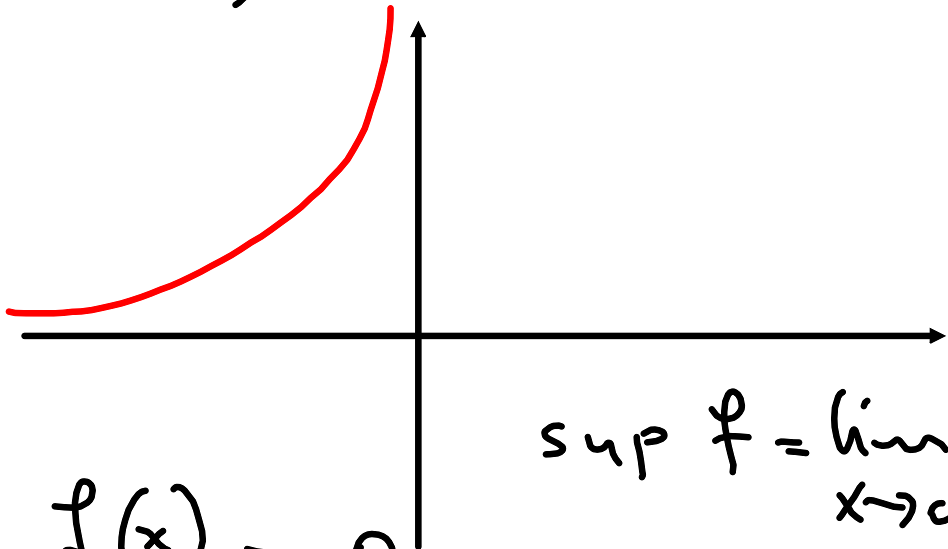
Prop: Se $f:(a,b)\to\mathbb{R}$ è debolmente crescente allora

$$\exists \lim_{x\to b^-} f(x) = \sup_{(a,b)} f$$

$$\exists \lim_{x\to a^+} f(x) = \inf_{(a,b)} f$$

Es: $f(x) = -\frac{1}{x}$

$f : (-\infty, 0) \longrightarrow \mathbb{R}$



$\inf_{(-\infty,0)} f = \lim_{x \to -\infty} f(x) = 0$

$\sup f = \lim_{x \to 0^-} f(x) = +\infty$

Oss: $\lim_{x \to x_0} f(x) = 0$

se e solo se

$$\lim_{x \to x_0} |f(x)| = 0$$

Es: Sia

$$A = \left\{ x \in \mathbb{R} : \frac{e^{3x^3} \cdot e^3}{e^{2x}} > e^{2x^2+x+1} \right\}$$

dire se A è inferiormente o superiormente limitato.

$$\frac{e^{3x^3} \cdot e^3}{e^{2x}} = e^{3x^3+3-2x}$$

$$e^{3x^3-2x+3} > e^{2x^2+x+1}$$

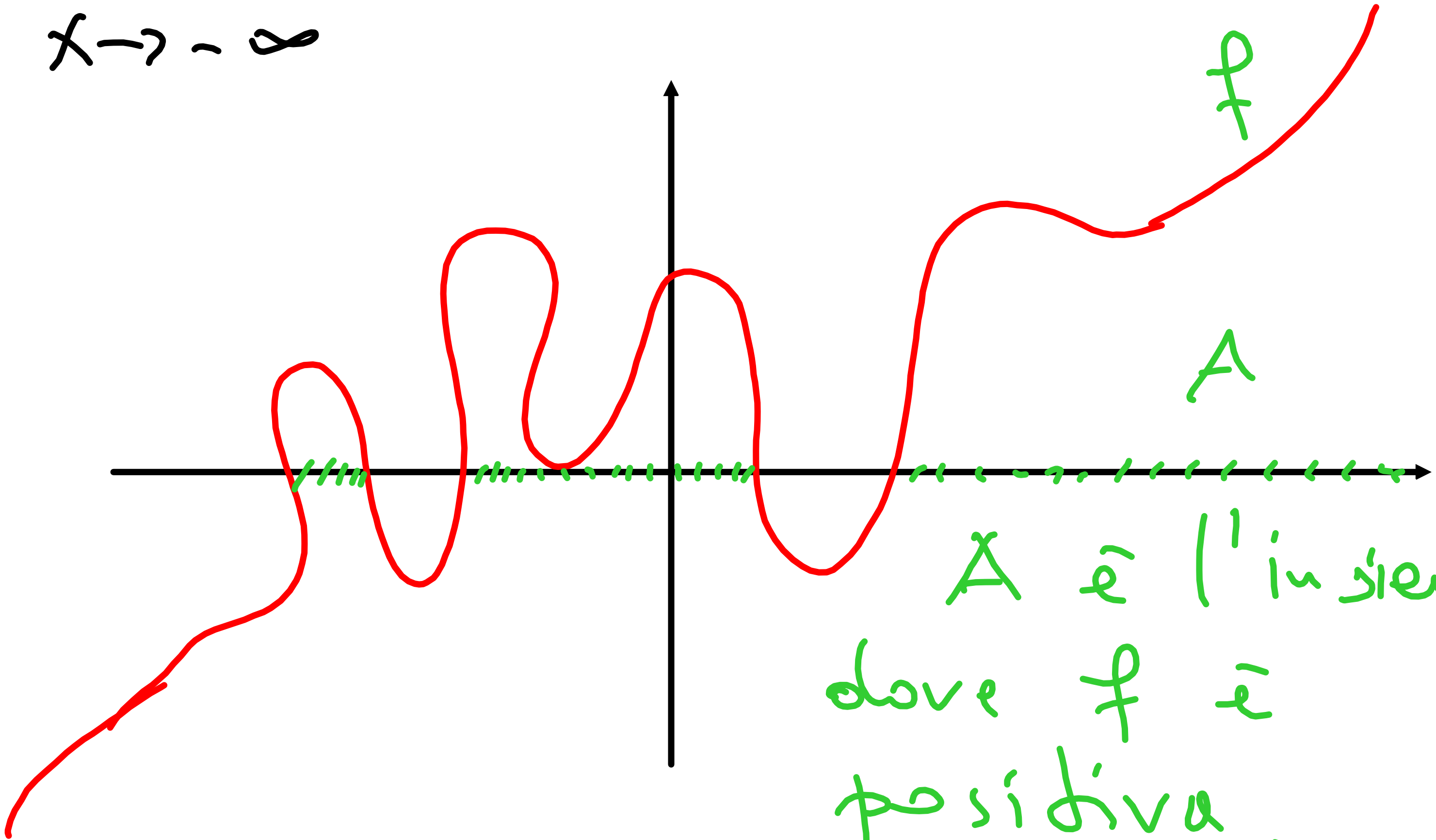
l'esponenziale è crescente

$$\Rightarrow 3x^3-2x+3 > 2x^2+x+1$$

$$3x^3-2x^2-3x+2 > 0 \quad *$$

$$\lim_{x\to+\infty} 3x^3-2x^2-3x+2 = +\infty$$

$$\lim_{x \to -\infty} 3x^3 - 2x^2 - 3x + 2 = -\infty$$



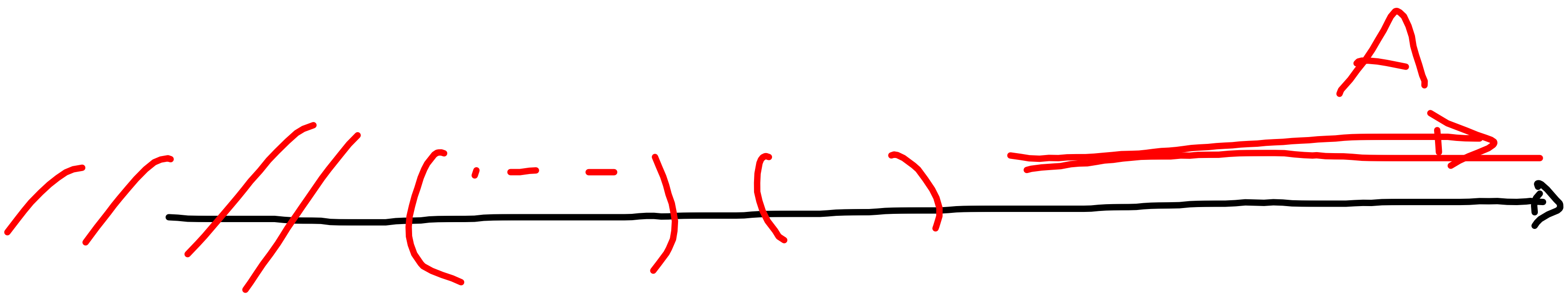
A è l'insieme dove f è positiva.

Se x è abbastanza grande $f(x)>0$ e x è abbastanza piccolo $f(x)<0$ (permanenza del segno).

quindi se x è abbastanza grande (*) vale

$\Rightarrow x \in A$ se x è abbastanza piccolo $\Rightarrow$ (*) non vale

e $x \notin A$.



quindi A è inferiormente limitato ma non superiorm. limitato.