Integrazione per parti

$f, g : I \longrightarrow \mathbb{R}$, I intervallo

f continua , $g \in C^1$.

Sia F una primitiva di f

allora

$$\int f g \, dx = F g - \int F g' \, dx \, .$$

dim: $(Fg)' = F'g + Fg' =$

$= fg + Fg'$

integro

$$\int (Fg)'\,dx = \int fg\,dx + \int Fg'\,dx$$

$$Fg = \int fg\,dx + \int Fg'\,dx$$

$$\int fg\,dx = Fg - \int Fg'\,dx \qquad \square$$

$\underline{\text{Es}}$: $\int \underset{g}{x} \, \underset{f}{\sin x} \, dx =$

$g' = 1$, $F = -\cos x$

$= -\cos x \cdot x - \int -\cos x \cdot 1 \, dx =$

$= -x \cos x + \sin x + c$.

Es: $\int \log x \, dx =$

$$= \int \underset{f}{1} \cdot \underset{g}{\log x} \, dx = \qquad F = x \quad g' = \frac{1}{x}$$

$$= x \cdot \log x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx =$$

$$= x \log x - x + c \, .$$

Es: $\boxed{\int \cos^2 x \, dx} =$

$= \int \underbrace{\cos x}_{f} \; \underbrace{\cos x}_{g} \; dx$

$F = \sin x \qquad g' = -\sin x$

$$= \sin x \cdot \cos x - \int \sin x \, (-\sin x) \, dx$$

$$= \sin x \cos x + \int \sin^2 x \, dx =$$

$$= \sin x \cos x + \int 1 - \cos^2 x \, dx =$$

$$= \boxed{\sin x \cos x + x - \int \cos^2 x \, dx}$$

$$\int \cos^2 x \, dx = \sin x \cos x + x - \int \cos^2 x \, dx$$

$$2 \int \cos^2 x \, dx = \sin x \cos x + x + c$$

$$\int \cos^2 x \, dx = \frac{\sin x \cos x + x}{2} + c \, .$$

Oss: $D\left[\log f(x)\right]$ $\qquad \underline{f > 0}$

$$= \frac{1}{f(x)} \cdot f'(x)$$

Es: $\int \operatorname{arctg} x \, dx =$

$$= \int \underset{f}{1} \cdot \underset{g}{\operatorname{arctg} x} \, dx =$$

$$= x \operatorname{arctg} x - \int x \cdot \frac{1}{1+x^2} \, dx$$

$$= x \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$$

$$= x \operatorname{arctg} x - \frac{1}{2}\log(1+x^2) + c$$

Integrazione per sostituzione

I, J intervalli di $\mathbb{R}$

$f : I \to \mathbb{R}$ continua

$\varphi : J \to I$ di classe C^1

allora, se F è una primitiva di f

$$\int (f \circ \varphi)\, \varphi' \, dx = (F \circ \varphi) + c$$

$$= \int f(t)\,dt \Big|_{t=\varphi(x)}$$

dim: $(F \circ \varphi)' = (F' \circ \varphi) \cdot \varphi'$

$= (f \circ \varphi)\, \varphi'$ integrate

$$\Rightarrow F \circ \varphi = \int (f \circ \varphi)\, \varphi' \, dx \quad . \quad \square$$

__Es.__ $\int x e^{x^2} dx =$

$\varphi(x) = x^2 \qquad \varphi'(x) = 2x, \; x = \frac{\varphi'}{2}$

$= \int \frac{\varphi'(x)}{2} e^{\varphi(x)} dx =$

poniamo $f(t) = \frac{e^t}{2}$

$= \int f(\varphi(x)) \varphi'(x) dx$

F primitiva di f

$$F(t) = \int \frac{e^t}{2}\, dt = \frac{1}{2} e^t$$

$$\Rightarrow \int (f \circ \varphi)\, \varphi'(x)\, dx = (F \circ \varphi) + c$$

$$= \frac{1}{2} e^{\varphi(x)} + c = \frac{1}{2} e^{x^2} + c .$$

verifica $D\left(\frac{1}{2} e^{x^2}\right) = \frac{1}{\cancel{2}} \cdot \cancel{2}x\, e^{x^2}$.

Metodo pratico per la sostituzione.

$$\int x e^{x^2} dx$$ pongo $x^2 = t = t(x)$

$$\Rightarrow \frac{dt}{dx} = 2x \Rightarrow dt = 2x\,dx$$

$$x\,dx = \frac{dt}{2}$$

sostituisco nell'integrale compreso nel dx

$$\int x e^{x^2} dx = \int e^t \frac{dt}{2} =$$

$$= \frac{e^t}{2} + c \underset{\substack{\uparrow \\ t = x^2}}{=} \frac{e^{x^2}}{2} + c$$

Es: $\int \frac{tg\, x}{\cos^2 x}\, dx =$

$= \int \frac{\sin x}{\cos^3 x}\, dx$ $\qquad \cos x = t$

$\frac{dt}{dx} = -\sin x$

$\Rightarrow \sin x\, dx = -dt$

$= \int \frac{-dt}{t^3} = -\int t^{-3}\, dt =$

$$= -\left(-\frac{1}{2}\right) t^{-2} + c = \frac{1}{2t^2} + c =$$

$$= \frac{1}{2\cos^2 x} + c \qquad t = \cos x$$

Es: $\int \sqrt{1-x^2}\, dx =$

$x = \sin t \qquad \frac{dx}{dt} = \cos t$

$dx = \cos t\, dt$

$= \int \sqrt{1-\sin^2 t} \cdot \cos t\, dt$

$= \int \sqrt{\cos^2 t}\, \cos t\, dt =$

$$= \int |\cos t| \cos t \, dt$$

supponiamo che sia $\cos t > 0$

$$= \int \cos^2 t \, dt =$$

$$\frac{t + \sin t \cos t}{2} + c =$$

$x = \sin t \Rightarrow t = \arcsin x$

$$= \frac{\arcsin x + \sin(\arcsin x)\cos(\arcsin x)}{2} + c$$

$$= \frac{\arcsin x + x\cos(\arcsin x)}{2} + c$$

$$= \circledast$$

$$\cos t = \sqrt{1 - \sin^2 t}$$

$$\cos(\arcsin x) = \sqrt{1 - \sin^2(\arcsin x)}$$

$$= \sqrt{1 - x^2}$$

$$= \frac{\arcsin x + x\sqrt{1-x^2}}{2} + C.$$

Es: $D(\arcsin)$

$f(x) = \sin x$, $f^{-1}(y) = \arcsin y$

$$D(f^{-1}(y)) = \frac{1}{f'(f^{-1}(y))} =$$

$f'(x) = \cos x$

$$= \frac{1}{\cos(f^{-1}(y))} = \frac{1}{\cos(\arcsin y)} =$$

$$= \frac{1}{\sqrt{1-\sin^2(\arcsin y)}} = \frac{1}{\sqrt{1-y^2}}.$$

quindi

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c.$$

Es: $D(\arccos x) =$

$$= \frac{-1}{\sqrt{1-x^2}} .$$

Oss: $D(\arcsin x + \arccos x)$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

$\Rightarrow \arcsin x + \arccos x =$ costante

quanto vale la costante?

la calcolo per $x = 0$.

costante = arcsin 0 + arccos 0 =

$$= 0 + \frac{\pi}{2} = \frac{\pi}{2} .$$

$$\arcsin x + \arccos x = \frac{\pi}{2} .$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$$

$$= \frac{\pi}{2} - \arccos x + c =$$

$$= - \arccos x + d$$

Integrazione di funzioni razionali con denominatore di 2° grado.

$$\int \frac{ax+b}{x^2+cx+d}\,dx$$

(caso 1) $\displaystyle\int \frac{dx}{(x-a)^2}$

$x-a=t \qquad dx=dt$

$$= \int \frac{dt}{t^2} = \int t^{-2}\,dt = -t^{-1} + c$$

$$= \frac{-1}{x-a} + c$$

2) $$\int \frac{dx}{(x-a)(x-b)} \quad \text{con } a \neq b.$$

cerco di scrivere

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

determino A e B.

$$\frac{A}{x-a} + \frac{B}{x-b} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)} =$$

$$= \frac{Ax-Ab+Bx-Ba}{(x-a)(x-b)} =$$

$$= \frac{\overset{0}{\boxed{(A+B)}}x \overset{1}{\overbrace{-Ab-Ba}}}{(x-a)(x-b)} = \frac{0\cdot x+1}{(x-a)(x-b)}$$

$$\begin{cases} A + B = 0 \\ -Ab - Ba = 1 \end{cases}$$

$B = -A$

$-Ab + Aa = 1$

$\Downarrow$

$A(a-b) = 1$

$$B = \frac{1}{b-a}$$

$$A = \frac{1}{a-b}$$

$a \neq b.$

$$\frac{1}{(x-a)(x-b)} = \frac{\frac{1}{a-b}}{x-a} + \frac{\frac{1}{b-a}}{x-b} =$$

$$= \frac{1}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right)$$

$$\Rightarrow \int \frac{dx}{(x-a)(x-b)} =$$

$$= \frac{1}{a-b} \int \frac{1}{x-a} - \frac{1}{x-b}\, dx =$$

$$= \frac{1}{a-b}\left(\int \frac{dx}{x-a} - \int \frac{dx}{x-b}\right) =$$

$$= \frac{1}{a-b}\left(\log|x-a| - \log|x-b|\right) + c$$

$$= \frac{1}{a-b}\log\left|\frac{x-a}{x-b}\right| + c$$

$$\int \frac{dx}{x^2-8x+15}$$

$$x^2-8x+15=0 \iff x=4\pm\sqrt{16-15}=\begin{cases}5\\3\end{cases}$$

$$\int \frac{dx}{(x-5)(x-3)} = \qquad \boxed{a=5\ ,\ b=3}$$

$$=\frac{1}{5-3}\log\left|\frac{x-5}{x-3}\right|+c=\frac{1}{2}\log\left|\frac{x-5}{x-3}\right|+c$$