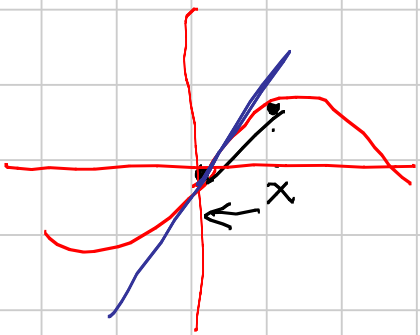
NOTE ES. 2/4 ottobre 19 (4 ottobre)

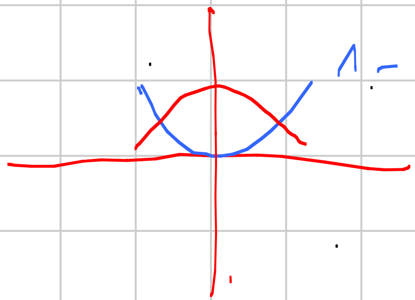
Note Title 10/4/2019

1) LIMITI NOTEVOLI TRIGONOMETRICI

$x \mapsto \sin x$, $\cos x$ sono continue

$\exists \lim_{x \to 0} \frac{\sin x}{x} = 1$ $\qquad$ $\exists \lim_{x \to 0} \frac{1 - \cos x}{\frac{x^2}{2}} = 1$





2) $\underline{a \in \mathbb{R}}$, $x > 0$: x^a $\qquad \lim_{x \to 0^+} x^a$, $\lim_{x \to +\infty} x^a$

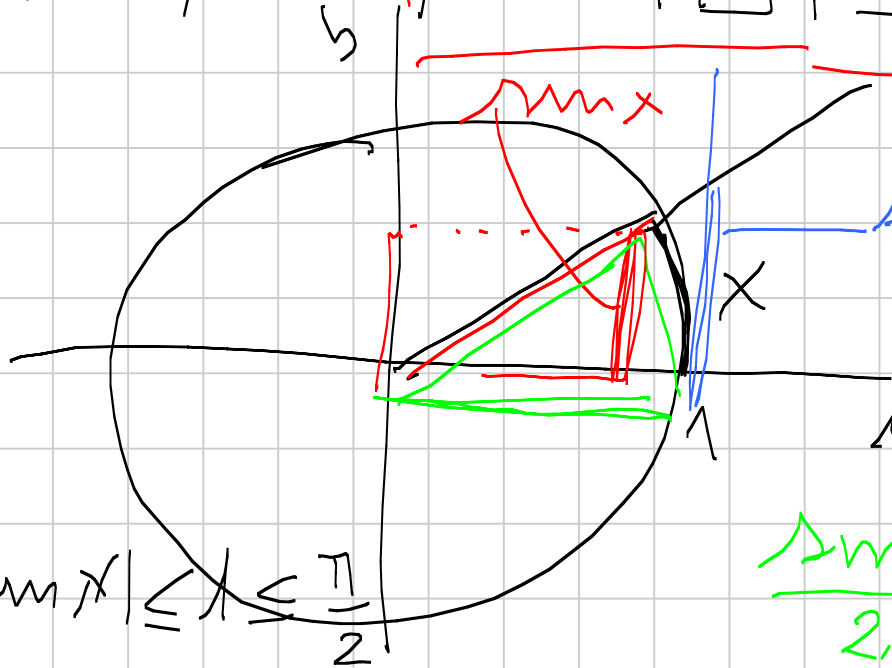
1) INGREDIENTI

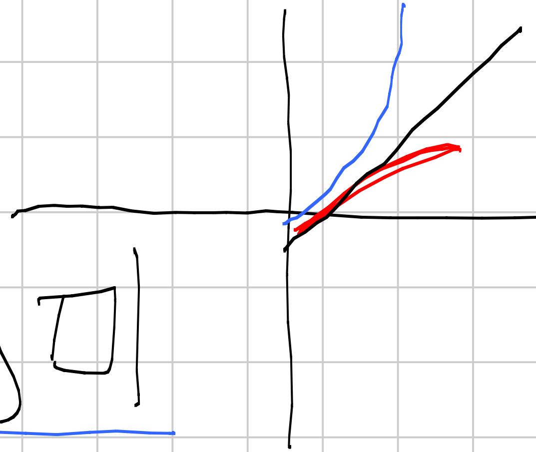
1.1)

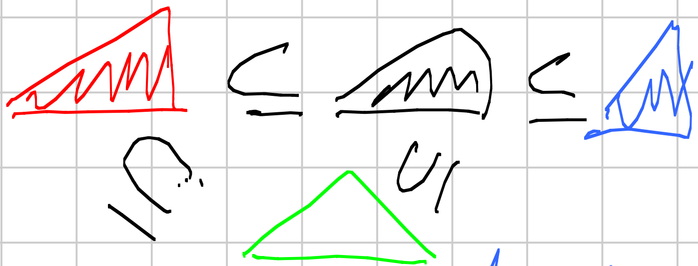
$$\boxed{|\sin \square| \le |\square|} \le |\text{tg}\, \square|$$

$|x| \le \frac{\pi}{2}$

per simm. $0 \le x < \frac{\pi}{2}$







$$\frac{\sin x}{2} \le \frac{x}{2} \le \frac{\text{tg}\, x}{2}$$

$|\sin x| \le 1 < \frac{\pi}{2}$

$$\sin x \le x \le \text{tg}\, x \quad \text{per } 0 \le x < \frac{\pi}{2}$$

- $|\overset{\cos}{\sin}\alpha - \overset{\cos}{\sin}\beta| \leq |\alpha - \beta|$

$$\left|\frac{\sin\alpha - \sin\beta}{\alpha - \beta}\right| \leq 1$$

FORMULE DI PROSTAFERESI

$$\sin a + \underset{\sin(-\beta)}{\sin b} = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$$

$a = \alpha$
$b = -\beta$

$$\boxed{|\sin\alpha - \sin\beta|} = 2\left|\sin\frac{\alpha-\beta}{2}\right|\left|\cos\frac{\alpha+\beta}{2}\right| \leq$$

$$\leq \cancel{2}\left|\frac{\alpha-\beta}{\cancel{2}}\right|\left|\cos\frac{\alpha+\beta}{2}\right| = |\alpha-\beta| \cdot \left|\cos\frac{\alpha+\beta}{2}\right| \leq \boxed{|\alpha-\beta|}$$

$f : D \longrightarrow \mathbb{R}$ $\quad x_0 \in [-\infty, +\infty]$ di accum. per D

$\exists \lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$

$\begin{cases} x \in D \\ x \neq x_0 \end{cases}$

$\forall \varepsilon > 0 \ \exists J$ intorno di x_0

$\forall x \in \underline{J \cap D \setminus \{x_0\}} \neq \emptyset$

$|f(x) - \ell| \leq \varepsilon$

$\boxed{d} = |\ell_1 - \ell_2| = |\ell_1 - f(x) + f(x) - \ell_2|$

$\leq |\ell_1 - f(x)| + |f(x) - \ell_2| \leq \boxed{\frac{2}{3} d}$

$\varepsilon = \frac{d}{3}$

$x \in J_1 \cap J_2 \cap D \setminus \{x_0\}$

① il limite è unico

$\ell_1 \quad d \quad \ell_2 \quad x_0 \quad J_1$

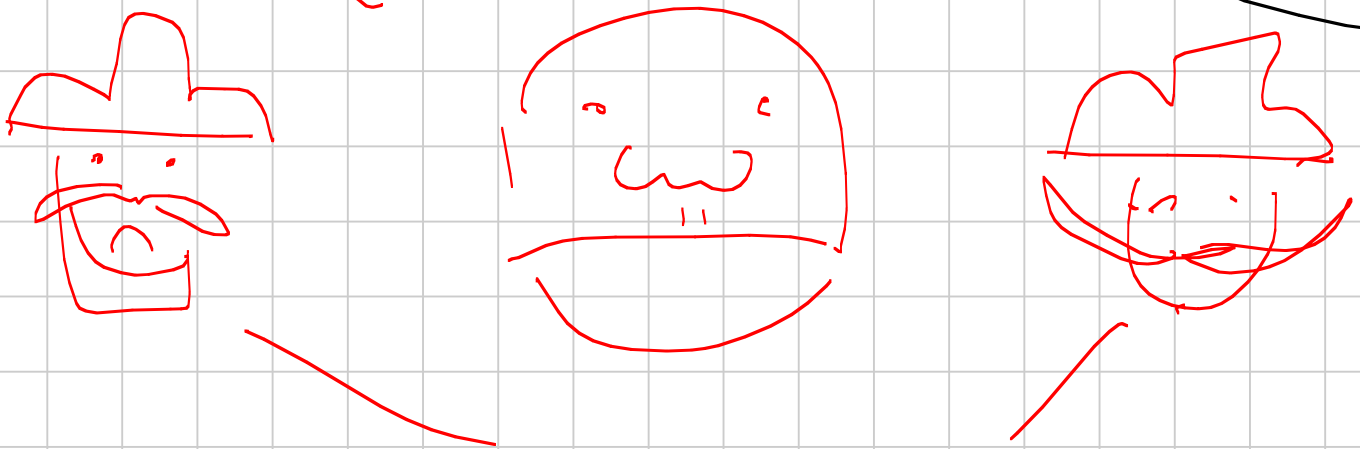
② teorema dei Carabinieri

$$g(x) \leq f(x) \leq h(x) \qquad x \in I \cap D \setminus \{x_0\}$$

$$\lim_{x \to x_0} g(x) = \ell \qquad \lim_{x \to x_0} h(x) = \ell$$

allora

$$\exists \lim_{x \to x_0} f(x) = \ell$$



POSSIAMO ORA DIMOSTRARE I LIMITI TRIG.

- cont sin e cos $0 \le \underbrace{|\sin x - \sin x_0|}_{\downarrow\, 0} \le \underbrace{|x - x_0|}_{\downarrow\, 0}$

$\ell \in \mathbb{R}$

$$\boxed{\lim_{x \to x_0} f(x) = \ell \iff \lim_{x \to x_0} |f(x) - \ell| = 0}$$

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

$$|\sin x| \le |x| \le |\operatorname{tg} x|$$

$|x| \le \frac{\pi}{2}$

Dimostriamo

$$\exists \lim_{x \to 0^+} \frac{\sin x}{x} = 1$$

$|x| < \frac{\pi}{2}$

$$|\sin x| \le |x| \le |\operatorname{tg} x|$$

$x \to 0$

$x > 0$

$x > 0$ $\quad 0 < \sin x \le x \le \operatorname{tg} x$

$x > 0$ $\quad 0 < \sin x \le x \le \sin x \cdot \frac{1}{\cos x}$

$$1 = \cos 0 \xleftarrow[x \to 0^+]{} \cos x \le \frac{\sin x}{x} \le 1 \xrightarrow[x \to 0^+]{} 1$$

$$\exists \lim_{x \to 0} \frac{1-\cos x}{\frac{x^2}{2}} = 1$$

voglio usare $\frac{\sin \vartheta}{\vartheta} \xrightarrow[\vartheta \to 0]{} 1$

$$\frac{1-\cos x}{x^2} = \frac{(1-\cos x)(1+\cos x)}{x^2} \frac{1}{1+\cos x} =$$

$$(A-B)(A+B) = A^2 - B^2$$

$$= \frac{1-(\cos x)^2}{x^2} \frac{1}{1+\cos x} = \underbrace{\left(\frac{\sin x}{x}\right)^2}_{\downarrow\, 1} \cdot \underbrace{\frac{1}{1+\cos x}}_{\downarrow\, 1/2}$$

ELENCO LIMITI NOTEVOLI

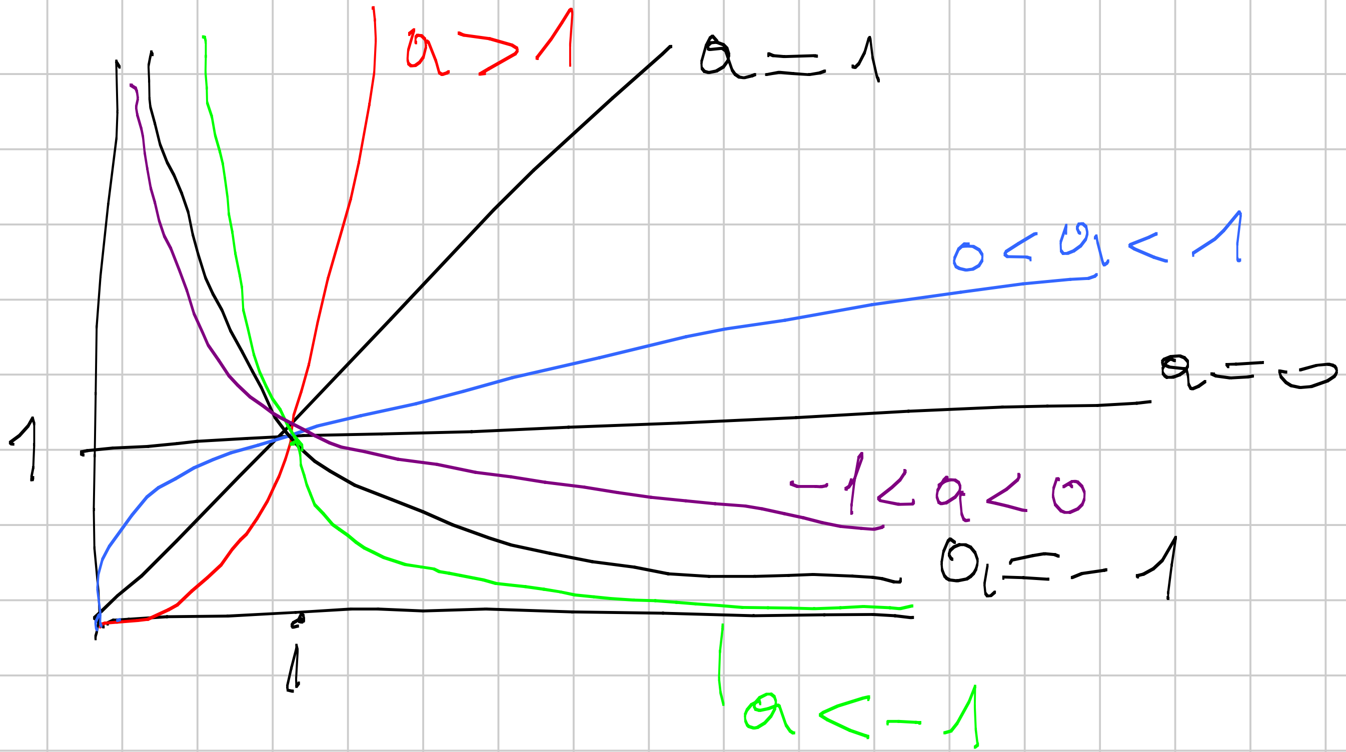
POTENZE ESP. REALE

$a \in \mathbb{R} \quad x > 0$

$$\lim_{x \to 0^+} x^a = \begin{cases} 0, & a > 0 \\ 1, & a = 0 \\ +\infty & a < 0 \end{cases} \qquad \lim_{x \to +\infty} x^a = \begin{cases} +\infty & a > 0 \\ 1, & a = 0 \\ 0, & a < 0 \end{cases}$$

NOTA

$a = -1 \quad x^{-1} \quad \lim_{0^+} \frac{1}{x} = +\infty \quad \lim_{0^-} \frac{1}{x} = -\infty \quad \nexists \lim_{x \to 0} \frac{1}{x}$



$\frac{1}{x^2}$ $\frac{1}{x}$

Esercizi simul. III sett.

D1 $\quad a = 0 \quad f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$

$f(0) = 0$

? $\lim_{x \to 0} f(x) = f(0)$ $\quad$ tra $a = 0$ $\nexists \lim_{0} f$

$a > 0 \quad ?\exists \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^a + a =$

$= a$

$?\exists \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} a e^{ax} =$

$= a \cdot e^{a \cdot 0} = a \, .$

l'esponenziale è continuo

D 2 $\lim_{x \to +\infty} \arcsin\left(\cos\left[\pi\left([x] + \frac{1}{x}\right)\right]\right)$

dom arcsin $= [-1, 1]$

arcsin è continua e iniettiva

se $\exists \lim_{+\infty} \cos\left(\pi[x] + \frac{\pi}{x}\right) = l$ $\quad$ $\lim_{+\infty} \arcsin \dots = \arcsin l$

? $\exists \lim\limits_{x \to +\infty} \cos\left(\pi[x] + \dfrac{\pi}{x}\right)$

$$\cos\left(\pi[x] + \frac{\pi}{x}\right) = \cos([x]\cdot\pi)\cos\frac{\pi}{x} - \cancel{\sin([x]\pi)}\sin\frac{\pi}{x}$$

$$= \cos([x]\cdot\pi)\cdot\cos\frac{\pi}{x} = \boxed{(-1)^{[x]}\cdot\cos\frac{\pi}{x}} \qquad \sin([x]\cdot\pi) = 0$$

$$\cos\frac{\pi}{x} \xrightarrow[x \to +\infty]{} 1$$

$\nexists$ il limite $\quad x$: $[x]$ pari $\longrightarrow \frac{\pi}{2}$

$[x]$ dispari $\longrightarrow -\pi/2$

D 3 $\displaystyle \lim_{x\to 0} \frac{x^{11} - 3x^2 + \underbrace{\sin x}}{1-\cos x}$

$\dfrac{\sin x}{x} \to 1$ $\qquad\qquad$ $\dfrac{1-\cos x}{\frac{x^2}{2}} \to 1$

$\sin x = x + r$, $\dfrac{r}{x} \xrightarrow[x\to 0]{} 0$ $\qquad$ $1-\cos x = \dfrac{x^2}{2} + r$ $\quad \dfrac{r}{x^2} \to 0$

$$\frac{x^{11} - 3x^2 + \sin x}{1-\cos x} = \overset{2}{\frac{x^2}{1-\cos x}} \cdot \frac{x^{11} - 3x^2 + \sin x}{x^2} = 1$$

$$\frac{x^2}{1-\cos x}\left(\frac{x^{11}}{x^2}-\frac{3x^2}{x^2}+\frac{\sin x}{x^2}\right)$$

$$\frac{x^2}{1-\cos x} \xrightarrow{x\to 0} 2$$

$$\frac{x^{11}}{x^2} \xrightarrow{x\to 0} 0 \qquad \frac{3x^2}{x^2} \to 3 \qquad \frac{\sin x}{x^2} \; ?$$

$$\frac{\sin x}{x}\cdot\frac{1}{x}$$

$$\frac{\sin x}{x} \to 1$$

$$\frac{x^{11} - 3x^2 + \sin x}{1 - \cos x} =$$

$$\frac{x^{11} - 3x^2 + \sqrt{x} + 2}{\frac{x^2}{2} + R}$$

$$\frac{R}{x} \to 0 \qquad \frac{R}{x^2} \to 0$$

$$\frac{x}{x^2} \cdot \frac{x^{10} - 3x + \frac{2}{x} + 1}{R/x^2 + 1} \to 1$$

$\to 0 \quad \to 0 \quad \to 0 \quad \to 1 \quad \to 1$

4

$$a^{1/3} - b^{1/3} = \frac{a - b}{a^{2/3} + a^{1/3} b^{1/3} + b^{2/3}}$$

$$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$$

$A = a^{1/3}$

$B = b^{1/3}$

$a = x^2 + 8x$

$b = x^2$

$$\sqrt[3]{8x} \cdot x^{\frac{1}{3}} \cdot \frac{x^{-\frac{1}{3}}}{x^{-\frac{1}{3}}} \quad x \to +\infty \quad 1$$

$$\frac{1}{(x^2+8x)^{2/3} + (x^2+8x)^{1/3} x^{2/3} + x^{4/3}}$$

$$\frac{\cancel{8x^{4/3}}}{} \quad \frac{1}{\cancel{x^{4/3}}} \quad \frac{1}{(1+8/x)^{2/3} + (1+8/x)^{1/3} + 1}$$

D 5, 6, 7, 8 ~~10~~ esercizio a casa