

# Lezione 08/04

CALCOLO DELLA FATTORIZZAZIONE LU DI A

STRUMENTO: MATRICI ELEMENTARI GAUSS

DEF: Sia  $E \in \mathbb{R}^{n \times n}$ . E si dice elementare.

Gauss e  $I_k$  con  $1 \leq k \leq n$  e  $v \in \mathbb{R}^n$

con  $v_1 = v_2 = \dots = v_k = 0$  tale che

$$I_k = I - v e_k^T = I - \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Esempio  $h=4$

$$k=1 \quad E = \begin{bmatrix} 1 & & & \\ -v_2 & 1 & & \\ -v_3 & & 1 & \\ -v_4 & & & 1 \end{bmatrix} = I - \begin{bmatrix} 0 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$k=2 \quad E = \begin{bmatrix} 1 & & & \\ & 1 & & \\ -v_3 & & 1 & \\ -v_4 & & & 1 \end{bmatrix} = I - \begin{bmatrix} 0 \\ 0 \\ v_3 \\ v_4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

$$k=3 \quad E = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -v_k \end{bmatrix}$$

$$k=4 \quad E = I$$

Matrici elementari di Gauss = matrici

triangolare inferiori con elementi diagonali = 1

Altri elementi non nulli in 1 colonna  
 sotto l'elemento diagonale.

### Proprietà

$$\textcircled{1} \quad E = I - v e_k^T \text{ è invertibile e}$$

$$E^{-1} = I + v e_k^T \quad \left( \begin{array}{l} \text{e ancora elementari} \\ \text{di Gauss} \end{array} \right)$$

Dim  $(I - v e_k^T)(I + v e_k^T) =$

$$I + \cancel{v e_k^T} - \cancel{v e_k^T} - v (e_k^T v e_k^T) \stackrel{=0}{=} I$$

(2) Sot.  $x \in \mathbb{R}^n$  con  $x_k \neq 0 \exists F \in \mathbb{R}^{n \times n}$   
 elementari di Gauss tale che

$$F x = \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ \vdots \\ 0 \end{bmatrix}$$

Dim: Cerchiamo  $F = I - v e_k^T$

abbiamo determinante 1. Vediamo che

$$\begin{aligned} (I - v e_k^T) x &= x - v (e_k^T x) \\ &= x - v x_k \end{aligned}$$

Implication of  $X - VX_k = \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$(\Rightarrow) VX_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{k+1} \\ \vdots \\ x_n \end{bmatrix}$   $\leftarrow$

$\Leftrightarrow \underline{v_1 = v_2 = \dots = v_k = 0} \quad \square$

$v_j x_k = x_j \quad j = k+1 : n$

$\underline{v_j = \frac{x_j}{x_k}} \quad j = k+1 : n \quad \square$

③ Il prodotto  $\bar{E}x$  con  $\bar{E}$  viene  
 calcolato grazie cost.  $O(n)$  ops

$\bar{E} = I - vev^T$

$\bar{E}x = (I - vev^T)x = X - VX_k$

$$z \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ kx_{k+1} - x_k \cdot v_{k+1} \\ \vdots \\ x_n - x_k \cdot v_n \end{bmatrix} \quad \begin{array}{l} k-k \text{ operasi} \\ \text{perkalian} \\ k-k \text{ operasi} \text{ skalar} \end{array}$$


---

### CALCULUS FORTKULAWAN LU

$$A = A^{(0)} = \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1n}^{(0)} \\ \vdots & & \vdots \\ a_{m1}^{(0)} & \dots & a_{nm}^{(0)} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

$\downarrow$  pers.

Suppose  $\underline{a_{11}^{(0)}} \neq 0$

$$I_1 \pm \frac{v_{e_1}^T}{a_{11}^{(0)}} \text{ to } I_2 \quad \begin{bmatrix} a_{11}^{(0)} \\ \vdots \\ a_{m1}^{(0)} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11}^{(0)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$V_{j2} \frac{a_{j1}^{(0)}}{a_{11}^{(0)}} \quad j=2 \dots m$$

(Multiplikation durch Gauss)

colada  $A^{(1)} = E_1 A^{(0)}$

$$A^{(1)} = \begin{bmatrix} 1 & & & \\ -v_2 & & & \\ \vdots & & & \\ -v_n & & & 1 \end{bmatrix} \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1n}^{(0)} \\ \vdots & & \vdots \\ a_{n1}^{(0)} & \dots & a_{nn}^{(0)} \end{bmatrix}$$

$$A^{(1)} = \begin{bmatrix} a_{11}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(1)} & \dots & a_{2n}^{(1)} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2}^{(1)} & \dots & a_{nn}^{(1)} \end{bmatrix}$$

$$a_{2i}^{(0)} = a_{ii}^{(1)} \quad (i=1, \dots, n)$$

(to  $I^0$  und von  $e_i$  modifiziert):

$$a_{i\tau}^{(1)} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ -v_i & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} a_{i1}^{(0)} \\ \vdots \\ a_{i\tau}^{(0)} \\ \vdots \\ a_{in}^{(0)} \end{bmatrix}$$

$$= a_{ij}^{(0)} - v_i r_{1j}^{(0)}$$

$$= a_{ij}^{(0)} - \left( \frac{a_{i1}^{(0)}}{a_{11}^{(0)}} \right) r_{1j}^{(0)}$$

$$\left| \begin{array}{l} a_{ij}^{(1)} = a_{ij}^{(0)} - \left( \frac{a_{i1}^{(0)}}{a_{11}^{(0)}} \right) r_{1j}^{(0)} \\ i = 2:m, j = 2:m \end{array} \right.$$

- Costs
- (a) dot product:  $O(m)$  ops
  - (b) assignments to  $A^{(1)}$   $O(m^2)$  ops  
 $(n-1)^2$  eq. multi.  
 ...

II<sup>o</sup> step removes a matrix element  
multi with second elem.

$$A^{(1)} = \begin{bmatrix} a_{11}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & \underbrace{a_{22}^{(1)}} & \dots & a_{2n}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & \dots & a_{nn}^{(1)} \end{bmatrix}$$

Suppose  $a_{kk}^{(k-1)} \neq 0$  ( $a_{kk}^{(k-1)}$  pivot)

PS3 determine  $F_2$  to

$$\underline{F_2} \begin{bmatrix} a_{12}^{(1)} \\ a_{22}^{(1)} \\ \vdots \\ a_{n2}^{(1)} \end{bmatrix} = \begin{bmatrix} a_{12}^{(1)} \\ a_{22}^{(1)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underline{F_2} \begin{bmatrix} 1 & & & \\ & -v & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} \swarrow & & & \\ & -v_3 & & \\ & & \ddots & \\ & & & -v_n \end{bmatrix}$$

$$V_{j2} = \frac{a_{j2}^{(1)}}{a_{22}^{(1)}}$$

$$j = 3 \dots n$$



$$A^{(2)} \approx I_2 \cdot A^{(1)}$$

$$= \begin{bmatrix} a_{11}^{(2)} & \dots & a_{1n}^{(2)} \\ 0 & a_{22}^{(2)} & \dots \\ \vdots & 0 & \dots \\ 0 & a_{n3}^{(2)} & \dots \end{bmatrix}$$

.....

In conclusion

$$\text{se } a_{kk}^{(k-1)} \neq 0 \quad k=1 \dots n-1$$

Posso determinare  $I_1 \dots I_{n-1}$

ho tutti gli elementi di Gauss ide di

$$I_{n-1} \dots I_1 A^{(1)} = A^{(n-1)} = U$$

Wsk die Transform reper

$$F_{n-1} \dots F_2 A = U$$

$$\Rightarrow A = \underbrace{\begin{pmatrix} F_1^{-1} & \dots & F_{n-1}^{-1} \end{pmatrix}}_L \cdot U$$

$$A = L \cdot U \quad \text{f. Form: LU(A)}$$

$$F_{\text{reper}} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 0 & 5 & 1 \end{bmatrix}$$

A an alle f. Formen  $LU$  (Form der Ordnung 3  
ist unicat)

$$\mathbb{F}_2 \begin{pmatrix} 1 & & \\ -2 & 1 & \\ 0 & & 1 \end{pmatrix}$$

$$\mathbb{F}_2 A \sim \begin{pmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 5 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{pmatrix}$$

$$= A^{(a)}$$

$$\mathbb{F}_2 \approx \begin{pmatrix} 2 & & \\ & 2 & \\ -5 & & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix} \approx \cup$$

$$\mathbb{F}_2 \mathbb{F}_2 A \approx \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix} \mathbb{F}_1^{-1} & \mathbb{F}_2^{-1} \end{pmatrix}}_L U$$

$$L = \underline{\underline{\begin{pmatrix} \mathbb{F}_1^{-1} & \mathbb{F}_2^{-1} \end{pmatrix}}} = \begin{pmatrix} 2 & & \\ 2 & 2 & \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 2 & \\ & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ 2 & 2 & \\ 0 & 5 & 2 \end{pmatrix}$$

L. e. f. d. d. i.  
 Multiplikation.  
 Block  $\times$  Block

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 0 & 5 & 1 \end{pmatrix}$$

LU d. A

Iptesi de lavoro  $a_{kk}^{(k-1)} \neq 0$

Iptesi pe  $\exists$  al uncati  $\det A_k \neq 0 \quad k=1 \dots n$

$\Rightarrow \det A_k \neq 0 \quad k=1 \dots n$

$\Rightarrow a_{kk}^{(k-1)} \neq 0$

$A_1 = \begin{bmatrix} a_{11}^{(0)} \\ \vdots \\ a_{1n}^{(0)} \end{bmatrix} \quad \det A_1 \neq 0 \Leftrightarrow a_{11}^{(0)} \neq 0$

$\Rightarrow$  Quali pso pu  $R \cdot I^0$  pso

$$A^{(1)} = \begin{bmatrix} a_{11}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(1)} & \dots \\ \vdots & \vdots & \vdots \\ 0 & a_{n2}^{(1)} & \dots \end{bmatrix} = I^1 \cdot \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1n}^{(0)} \\ \vdots & \vdots & \vdots \\ a_{n1}^{(0)} & \dots & a_{nn}^{(0)} \end{bmatrix}$$

$\Rightarrow$  Pa' le struttura' principi de tot' de d'lu 2

$$\begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} \\ 0 & a_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -r_2 & 1 \end{bmatrix} \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} \\ a_{21}^{(0)} & a_{22}^{(0)} \end{bmatrix}$$

Byndet  $\boxed{a_{11}^{(0)} a_{22}^{(1)} = \det A_2}$

&  $\det A_2 \neq 0 \Rightarrow a_{22}^{(1)} \neq 0 \Rightarrow$  piv  
second piv. . . .

### RISOLUZIONE DI UN SISTEMA LINEARE

$$Ax = b \quad (A \text{ invertibile})$$

$$\underline{A} = LU. \quad \underline{O(n^3)}$$

$$Ax = b \Leftrightarrow LUx = b \Leftrightarrow \begin{cases} Ly = b \\ \underline{Ux = y} \end{cases}$$

$\underline{O(n^2)}$

Passo anche applicare l'elemento  $\alpha$   
l'attuale elemento  $\alpha$  di  $A$  e  $b$

$$Ax = b \quad A^{(0)} x = b^{(0)}$$

$$E_{n-1} \cdot E_1 A^{(0)} x = E_{n-1} \cdot E_1 b^{(0)}$$

$i$ ° step detour  $E_i$

$$E_i A^{(0)} x = E_1 b^{(0)}$$

Sistema equivalente

$$A^{(1)} x = b^{(1)}$$

$$E_2 A^{(1)} x = E_2 b^{(1)}$$

$$A^{(2)} x = b^{(2)} \dots$$

$$\dots \cup X \in \mathbb{R}^{(n-1)}$$

Metodo de eliminação genérica per

resolver o sistema linear  $Ax = b$

$$I_{\text{simp}}: \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

$$I_2 = \begin{pmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\rightarrow I_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & -5 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} = y$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} x_1 + 0x_2 + x_3 = 1 \\ 2x_1 + x_2 + 3x_3 = 2 \\ 0x_1 + 1x_2 + x_3 = 1 \end{array}$$

Пример:

$A_c(a_{ij})$

$a_{ij} = \min(i, j)$

$$A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

① Dato  $A$  ammette fattorizzazione LU ed inverso  
 effusiva determinabile.

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 2 \\ 2 & - & - \\ 2 & - & - \end{pmatrix}$$

$I_0$  per  $I_{12}$

$$\begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & -1 & & & & \\ & -1 & & & & \\ & -1 & & & & \\ & -1 & & & & \\ & & & & & 0 \end{pmatrix}$$

$$I_1 \cdot A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

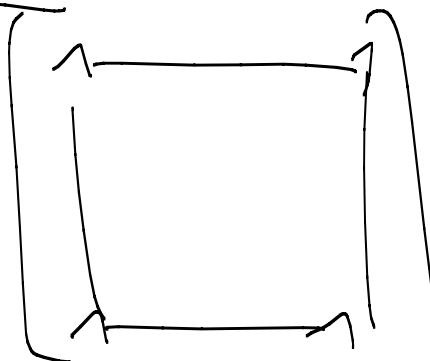
la matrice a cui devo applicare il metodo per  
 trovare LU è la stessa del 'po per questo  
 gli elementi diventano di 1

$$I_n = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

..... Il processo è applicabile  $\Leftrightarrow \exists LU$

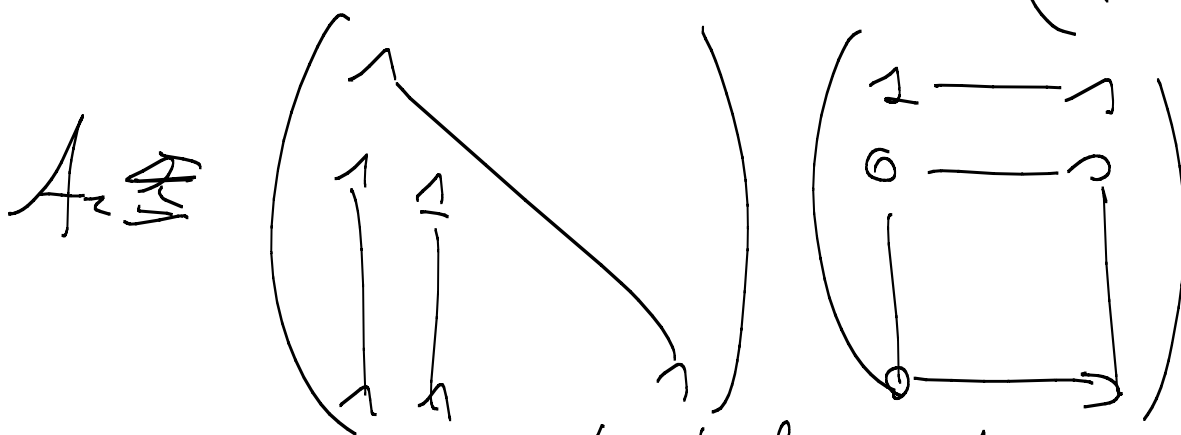
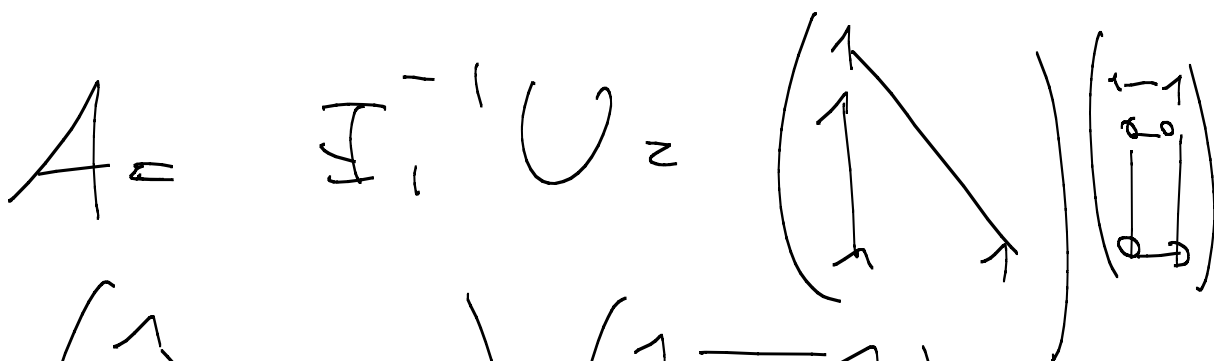
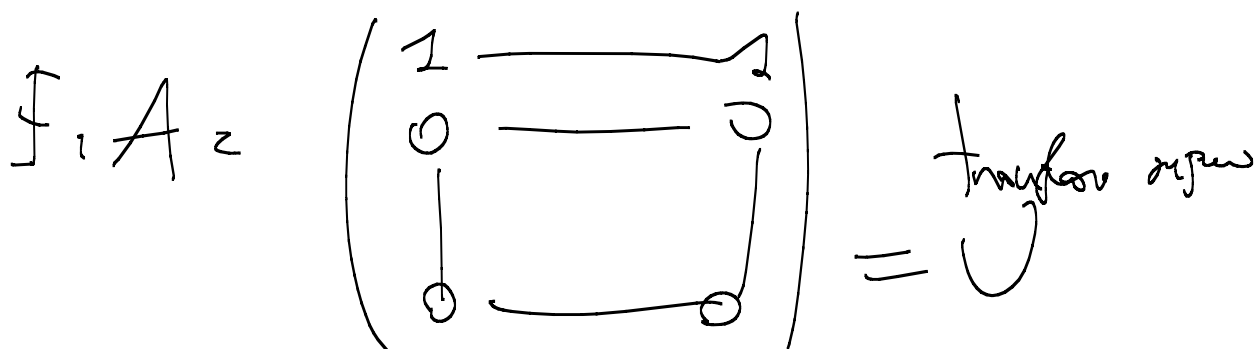
$$U_2 = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Processo  $\overline{A} =$  

- ① Dovere a A anche processo LU
- ② In caso offuscato decolorato e altri  
processo diverso e se punti.

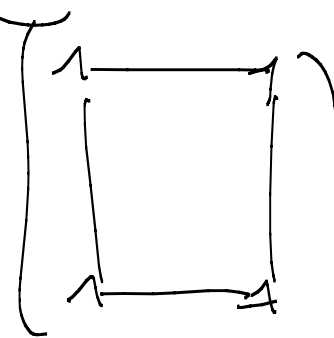
$$\det A_1 \neq 0 \quad \det A_2 = \det A_3 = \dots = \det A_n = 0$$



different from LU

Triangoli superiori:  $a_{ij} = 0$  se  $i > j$

→ Questo è l'autovalore di  $A \in \mathbb{R}^n$

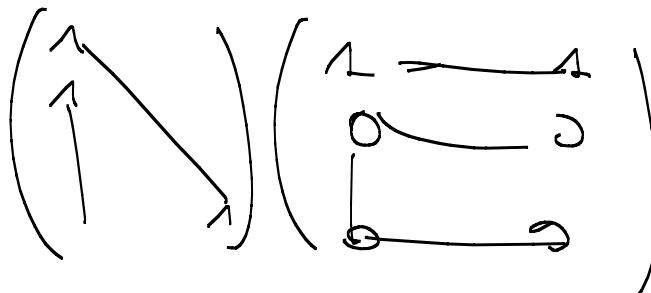


$A \subseteq \mathbb{R}^n$  det  $\neq 0$  set  $\mathbb{R}^n$

$A$  è invertibile  $(\Leftrightarrow) \mathbb{R}^n$  è funzione

$A$  ha un autovalore nullo  $(\Leftrightarrow) \mathbb{R}^n$  ha un autovettore

$A \in \mathbb{R}^n$



$A$  ha un autovalore nullo

$$\underline{Ax = 0} \Leftrightarrow \begin{cases} \mathbb{R}^n \setminus \{0\} \\ \cup \\ \{0\} \end{cases}$$

$$\Leftrightarrow \frac{\lambda_1 + \lambda_2 + \dots + \lambda_n = 0}{}$$

per scegliere  $n-1$  componenti libere e per  
determinare l'ultima da assegnare e  
dunque due k.e.  $n-1$

Alternativa.

$$\underbrace{\text{due k.e. } A} + \text{due k.e. } A \in \mathbb{R}^n$$

$$= 1 = 1$$

due k.e.  $n-1$

0 è autovettore di  $A$  con  $\lambda = 0$

Ci sono  $n$  autovettori non nulli

$$\begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = n \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\underline{\delta = n}$$

$$\underline{\sigma = \tau = 1}$$

