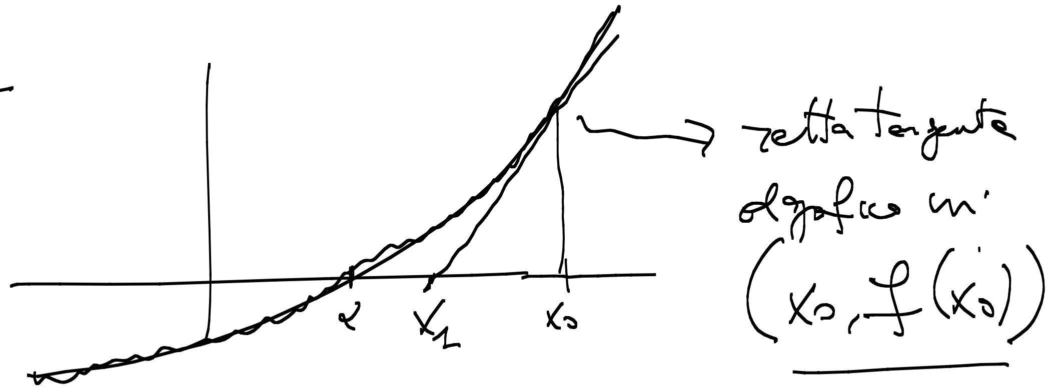
Lezione - 15/05

Metodo delle tangenti (Newton)

- Il più efficiente metodo di iterazione funzionale.
- semplicità di condizioni di convergenza. (espresse in termini di $f(x)$ e non $g(x)$)

Derivazione.



$$\begin{cases} y - f(x_0) = f'(x_0)\cdot(x - x_0) \\ y = 0 \end{cases}$$

$$\Rightarrow \quad x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Metodo delle tangenti.

$$\begin{cases} x_0 \in \mathbb{R} \\ x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)} \end{cases}$$

Metodo di iterazione funzionale $\quad x_{k+1} = g(x_k)$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Convergenza locale.

Teorema: Sia $f \in C^2([a,b])$ $f(\alpha) = 0$

$f'(\alpha) \neq 0$, $\alpha \in (a,b)$.

Allora il metodo delle tangenti è localmente convergente in α e la convergenza è almeno quadratica, cioè $\exists\, I_\alpha = [\alpha - \rho, \alpha + \rho]$ t.c. $\forall x_0 \in I_\alpha$

① $x_k \in I_\alpha$

② $\lim_{k \to +\infty} x_k = \alpha$

convergenza locale.

③ Se $x_k \neq \alpha \ \forall k$ $\lim \dfrac{|x_{k+1}-\alpha|}{|x_k-\alpha|^2} = C \in \mathbb{R}$.

Dim: $f'(\alpha) \neq 0 \Rightarrow \exists \hat{I}_\alpha$ tc $f'(x) \neq 0 \ \forall x \in \hat{I}_\alpha$

$g : \hat{I}_\alpha \to \mathbb{R}$ $\quad g(x) = x - \dfrac{f(x)}{f'(x)}$

$$g'(x) = 1 - \frac{f'(x)^2 - f(x) f''(x)}{f'(x)^2} = \frac{f(x) f''(x)}{f'(x)^2}.$$

$$g'(\alpha) = \frac{f(\alpha) f''(\alpha)}{f'(\alpha)^2} = 0$$

Corollario del teorema del punto fisso $\Rightarrow \exists I_\alpha \subseteq \hat{I}_\alpha$ tc si ha convergenza locale.

· Sia $x_0 \in I_\alpha$ e $x_k \neq \alpha \ \forall k$

$$0 = f(\alpha) = f(x_k) + f'(x_k)(\alpha - x_k) + \frac{f''(\xi_k)(\alpha - x_k)^2}{2}$$

$$x_k - \frac{f(x_k)}{f'(x_k)} = \alpha + \frac{f''(\xi_k)}{f'(x_k)} \frac{(\alpha - x_k)^2}{2}$$

$$x_{k+1} = \alpha + \frac{f''(\xi_k)}{f'(x_k)} \frac{(\alpha - x_k)^2}{2}$$

$$\frac{x_{k+1}-\alpha}{(\alpha - x_k)^2} = \frac{1}{2} \frac{f''(\xi_k)}{f'(x_k)}$$

$$\lim_{k\to+\infty} \frac{|x_{k+1}-\alpha|}{|\alpha - x_k|^2} = \frac{1}{2}\left|\frac{f''(\alpha)}{f'(\alpha)}\right| \in \mathbb{R} .$$

Convergenz. in larg.

Teorema: Sia $f \in C^2([a,b])$ $f(\alpha)=0$ $\alpha \in (a,b)$.

Se $\exists\ \delta > 0$ tc

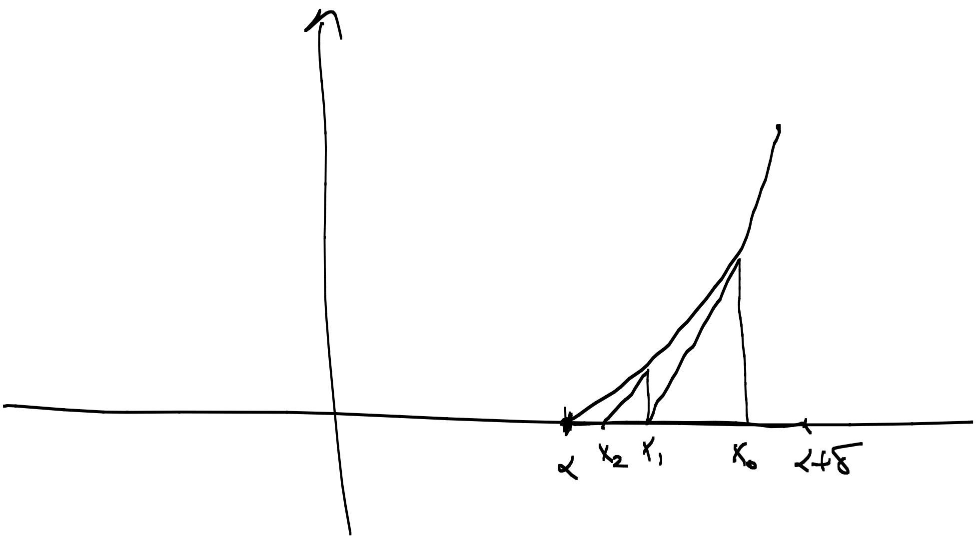
$$\begin{cases} f'(x) \neq 0 & \forall x \in (\alpha, \alpha+\delta] \subseteq [a,b] \\ f(x) f''(x) > 0 \end{cases}$$

allora $\forall x_0 \in (\alpha, \alpha+\delta]$ la successione generata dal metodo delle tangenti a partire da x_0 converge ad α

$f'(\cancel{x}) > 0$

$\Rightarrow f(x) > 0$

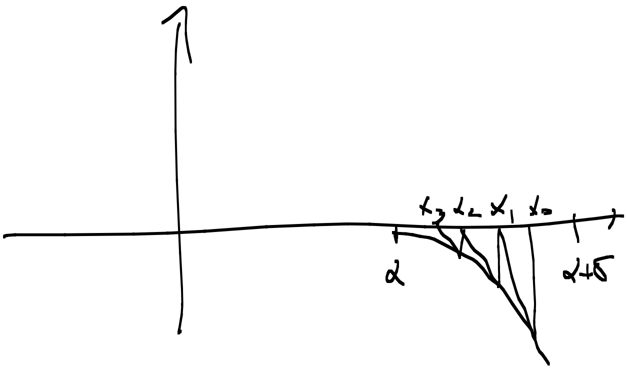
$\Rightarrow f''(x) > 0$



$f'(x) < 0$

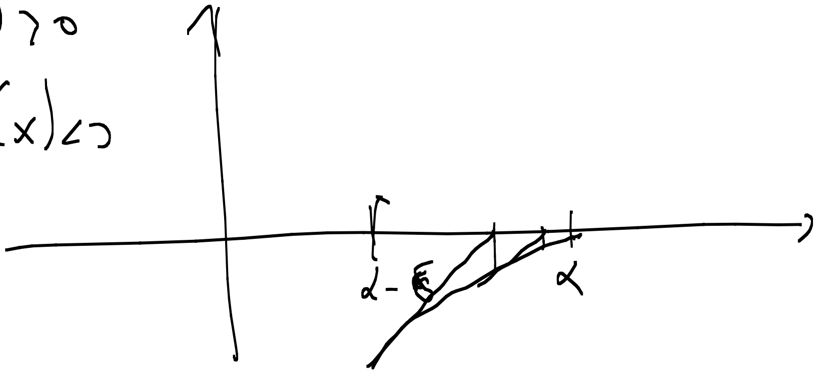
$\Rightarrow f(x) < 0$

$f''(x) < 0$



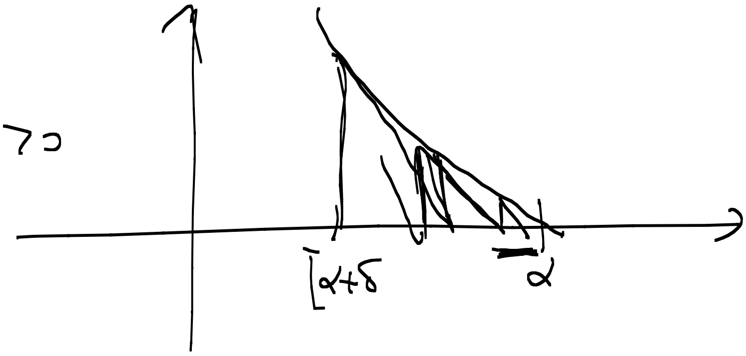
$f'(x) \neq 0 \quad f'(x) > 0$

$f(x) < 0 \Rightarrow f''(x) < 0$



$f'(x) < 0$

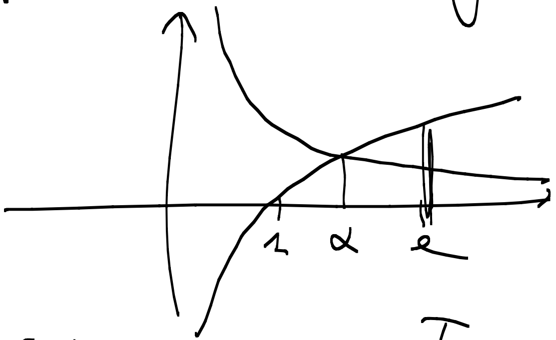
$f(x) > 0 \quad f''(x) > 0$



$f(x) = x \log x - 1 = 0$

$\underline{x > 0}$

$x \log x - 1 = 0 \iff x \log x = 1 \iff \log x = \frac{1}{x}$



$\exists! \; \alpha$ t.c. $f(\alpha) = 0$ $\qquad \underline{\alpha \in [1, e]}$

$f(x) = x \log x - 1$

$f \in C^\infty(\mathbb{R}^+)$

$\lim_{x \to 0^+} f(x) = ?$ $\qquad \lim_{x \to +\infty} f(x) = +\infty$

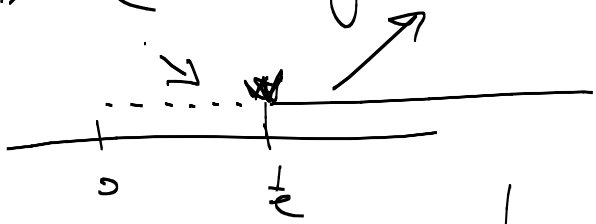
$$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{\frac{1}{x}} \overset{\text{Hopital}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

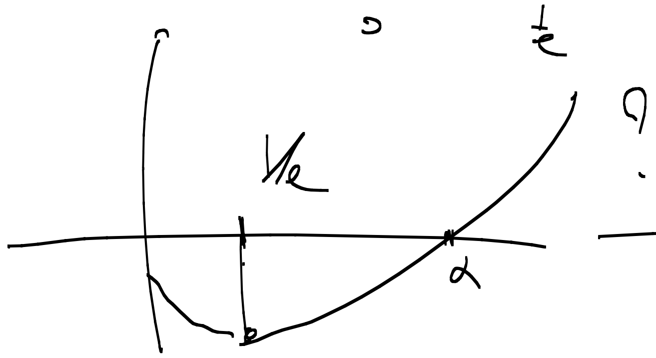
$$= \lim_{x\to 0^+} -\frac{1}{\cancel{x}}\cdot x^{\cancel{2}} = 0$$

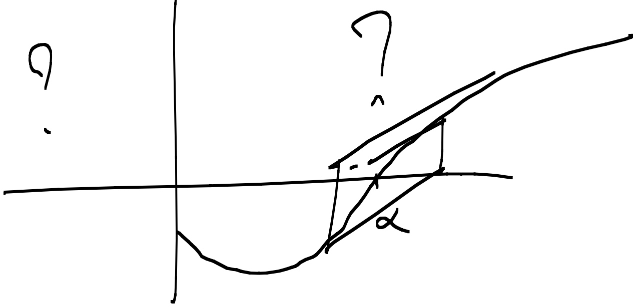
$$\lim_{x\to 0^+} f(x) = -1$$

$$f'(x) = 1\cdot \log x + x\cdot\frac{1}{x} = \log x + 1$$

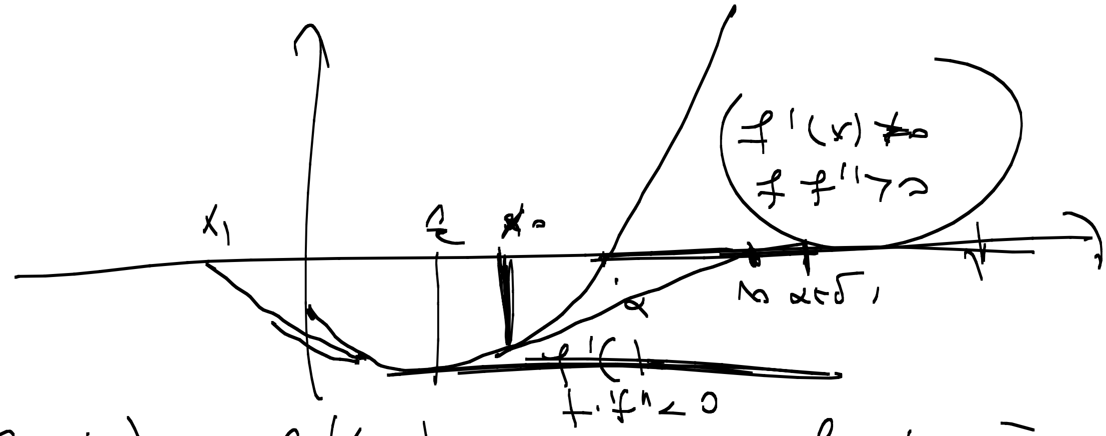
$$\log x + 1 \geq 0 \iff \log x \geq -1 \iff x \geq e^{-1} = \frac{1}{e}$$







$$f''(x) = \frac{1}{x} > 0 \quad \forall x > 0$$



$f \in C^2(\mathbb{R}^+)$ $\quad f'(\alpha) \neq 0 \implies$ il metodo è

localmente convergente in α

$\forall x_0 \geq \alpha$ il metodo delle tangenti genera ancora convergenza

---

Se $\frac{1}{e} < x_0 < \alpha \implies x_1 \geq \alpha \to$ convergenza.

$$x_1 - \alpha = g(x_0) - g(\alpha) =$$

$$\overset{\text{Lagrange}}{=} g'(\xi) \cdot (x_0 - \alpha) = \frac{f(\xi) f''(\xi)}{f'(\xi)^2}(x_0 - \alpha)$$

$\xi \in [x_0 \ \alpha]$

$x_0 - \alpha < 0 \qquad \dfrac{f''(\xi) f(\xi)}{(f'(\xi))^2} < 0$

$x_1 - \alpha \geq 0$ $\qquad$ $\underline{\underline{x_1 \geq \alpha}}$

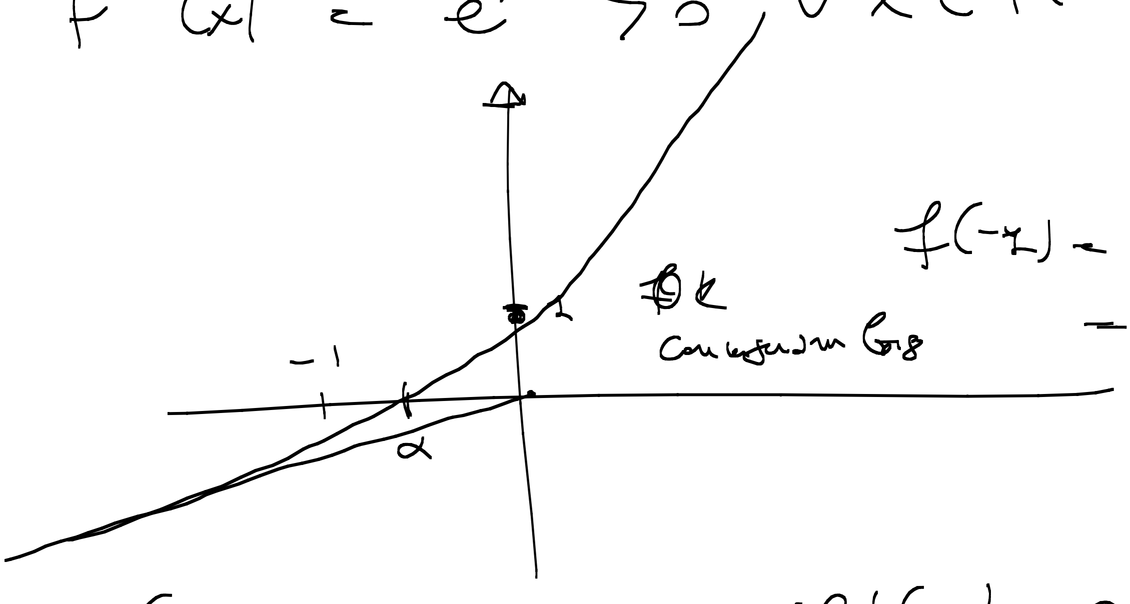
$f(x) = e^x + x = 0$

$f \in C^\infty(\mathbb{R})$ $\qquad \lim\limits_{x \to +\infty} f(x) = +\infty$

$\lim\limits_{x \to -\infty} f(x) = -\infty$

$f'(x) = e^x + 1 > 0 \quad \forall x \in \mathbb{R}$

$f''(x) = e^x > 0 \quad \forall x \in \mathbb{R}$



$f(-1) = e^{-1} - 1$

$= \frac{1}{e} - 1 < 0$

$\exists k$

Convergenza Globale

Convergenza locale $\quad$ f.i. $\quad f'(\alpha) = 0$

$\forall x_0 \in \mathbb{R} \qquad x_k \to \alpha$

$$f(x) = \frac{1}{x} - a = 0 \qquad x = \frac{1}{a} \quad (a \neq 0)$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k + \frac{\left(\frac{1}{x_k} - a\right)}{\frac{1}{x_k^2}}$$

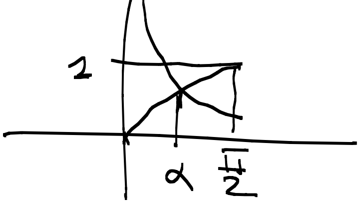
$$x_{k+1} = x_k + x_k^2 \cdot \left(\frac{1}{x_k} - a\right)$$

$$= x_k + x_k - a\,x_k^2$$

$$= 2x_k - a\,x_k^2 = x_k \cdot (2 - a\,x_k)$$

b/a $\qquad$ $\frac{1}{a}$ metodo delle tangenti $\qquad$ $b \cdot \frac{1}{a}$

$$f(x) = \sin x - \frac{1}{x}$$

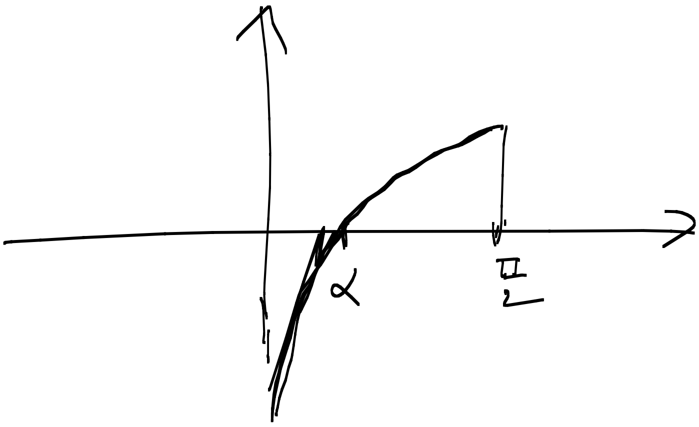


$\lim_{x \to 0^+} f(x) = -\infty$ $\qquad f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} - \frac{1}{\frac{\pi}{2}} = 1 - \frac{2}{\pi} > 0$

$f \in C^\infty(\mathbb{R}^+)$

$f'(x) = \cos x + \frac{1}{x^2} > 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right)$

$f''(x) = -\sin x - \frac{2x}{x^4} = -\sin x - \frac{2}{x^3}$
$< 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right)$



$\frac{\pi}{4}$?