Osservazione: se $f: A \to \mathbb{R}$, $C \subset A$, $x_0 \in Acc(C) \subset Acc(A)$

$$\exists \lim_{x \to x_0} f(x) = \ell \Rightarrow \exists \lim_{x \to x_0} g(x) = \ell \qquad g = f : C \to \mathbb{R}$$

$$(g = \underline{f|_C})$$

$$\swarrow$$

$$\forall V(\ell) \ \exists U(x_0) \ \forall x \in U(x_0) \cap \underset{\overset{\cup}{C}}{A} \setminus \{x_0\} \ f(x) \in V$$

Osservazione: se $x_0 \in Acc(C)$, $x_0 \in Acc(B)$, $B \cup C = A$, $f: A \to \mathbb{R}$

$$\exists \lim_{x_0} f|_C = \ell_1$$

$$\exists \lim_{x_0} f|_B = \ell_2 \qquad \text{e } \underline{\ell_1 = \ell_2} \iff \exists \lim_{x_0} f = \ell_1 = \ell_2 = \ell$$

$V(\ell)$ $\quad \exists U_1(x_0)$ $\underline{x \in U_1 \cap C \setminus \{x_0\} \ f(x) \in V}$ $\quad \exists U_2(x_0)$ $\underline{x \in U_2 \cap B \setminus \{x_0\} f(x) \in V}$

$U_1 \cap U_2$ è ancora intorno di x_0

Osservazione: per verificare che $\boxed{\nexists \lim_{x\to x_0} f(x)}$ $f: A \to \mathbb{R}$ $x_0 \in Acc(A)$ basta trovare quindi $C, B \subset A$, $x_0 \in Acc(C) \cap Acc(B)$ per cui

$$\exists \lim_{x_0} f|_C = l_1 \quad e \quad \exists \lim_{x_0} f|_B = l_2 \quad \boxed{\text{ma } l_1 \neq l_2}$$

Esempio: un altro modo per ottenere $\nexists \lim_{x\to+\infty} f(x)$ (cfr. pagg. 26-27 5 ott.) è anche

$f(x) = \sin x$, $A = \mathbb{R}$, $x_0 = +\infty$, $\underline{C = \left\{\frac{\pi}{2} + 2n\pi : n \in \mathbb{N}\right\}}$, $\underline{B = \left\{\frac{3}{2}\pi + 2n\pi : n \in \mathbb{N}\right\}}$

$$\exists \lim_{x\to+\infty} f|_C = \lim_{x\to+\infty} 1 = 1 \neq -1 = \lim_{x\to+\infty} -1 = \lim_{x\to+\infty} f|_B$$

— · —

Basandosi su quanto osservato e sul fatto che $f \circ g$ in sostanza restringe f ad $Im\, g$ si hanno i seguenti teoremi:

Teorema (di sostituzione, di cambio di variabile nei limiti o limite della composizione)

$A \xrightarrow{f} B \xrightarrow{g} \mathbb{R}$, $x_0 \in Acc(A)$. Se

$f(x) = y \xrightarrow[x\to x_0]{} y_0$

$?\ g \circ f \to y_0\ ?$

1) $\exists \lim_{x\to x_0} f(x) =: y_0$ 2) $y_0 \in Acc(B)$ 3) $\exists \lim_{y \to y_0} g(y) =: \ell$ e:

o 4.1) $y_0 \in B$ e g è continua in y_0 o 4.2) $\exists\, \mathcal{U}$ di x_0 $\forall x \in A \cap \mathcal{U} \setminus \{x_0\}$ $f(x) \neq y_0$

allora

$$\exists \lim_{x \to x_0} g(f(x)) = \ell = \lim_{y \to y_0} g(y)$$

Teorema (Ponte, di collegamento)

$f: A \to \mathbb{R}$, $x_0 \in Acc(A)$, si ha:

$\exists \lim_{x \to x_0} f(x) = \ell \iff$ per ogni $a_n : [m;\infty) \cap \mathbb{N} \to A$ tale che 1) $a_n \xrightarrow[n\to\infty]{} x_0$ e 2) $a_n \neq x_0$ si ha

$$\exists \lim_{n \to \infty} f(a_n) = \ell$$

$\Longrightarrow$

$*$ $g \to f$

$f \to a$

Limite della composizione di funzioni

Teorema: $A, B \subset \mathbb{R}$, $f: A \to B$, $g: B \to \mathbb{R}$, $x_0 \in Acc(A)$. Se esiste $\lim_{x \to x_0} f(x) = y_0$ e $y_0 \in Acc(B)$ e $\exists \lim_{y \to y_0} g(y) = \ell \in \overline{\mathbb{R}}$ e se è verificato almeno una delle 2 seguenti ipotesi:

1) $y_0 \in B$ e g è continua in y_0

2) esiste U intorno di x_0 t.c.

se $x \in U \cap A \setminus \{x_0\} \Rightarrow f(x) \neq y_0$

$y = f(x)$

Allora $\lim_{x \to x_0} (g \circ f)(x) = \ell$.

Cioè $\lim_{x \to x_0} (g \circ f)(x) = \lim_{y \to y_0} g(y)$.

Es: calcoliamo $\lim_{x \to -\infty} \operatorname{arctg}(x^2)$.

è una composizione

$f(x) = x^2 \qquad g(y) = \operatorname{arctg} y$

$(g \circ f)(x) = g(f(x)) = g(x^2) = \operatorname{arctg}(x^2)$

$x_0 = -\infty \quad , \quad y_0 = \underline{\lim_{x \to x_0} f(x)} = \lim_{x \to -\infty} x^2 = +\infty$

l'ipotesi 1) non è verificata perché $y_0 = +\infty$ e non appartiene al dominio di g

ma l'ipotesi 2) è ovviamente verificata perché chiedo che $f(x) \neq y_0$ cioè $f(x) \neq +\infty$ che è ovviamente sempre vero.

Applico il teorema

$$\lim_{y \to y_0} g(y) = \lim_{y \to +\infty} \operatorname{arctg} y = \frac{\pi}{2}$$

$$\Rightarrow \lim_{x \to -\infty} \operatorname{arctg}(x^2) = \frac{\pi}{2}.$$

Oss: è un teorema di cambiamento di variabile.

$$\lim_{\boxed{x}\to-\infty} \operatorname{arctg}\left(\boxed{x^2}\right) \qquad \Bigg| \qquad \lim_{y\to+\infty} \operatorname{arctg} y = \frac{\pi}{2}.$$

cambio variabile e pongo $\underline{y = x^2}$

se $x \to -\infty$ a quanto tende y?

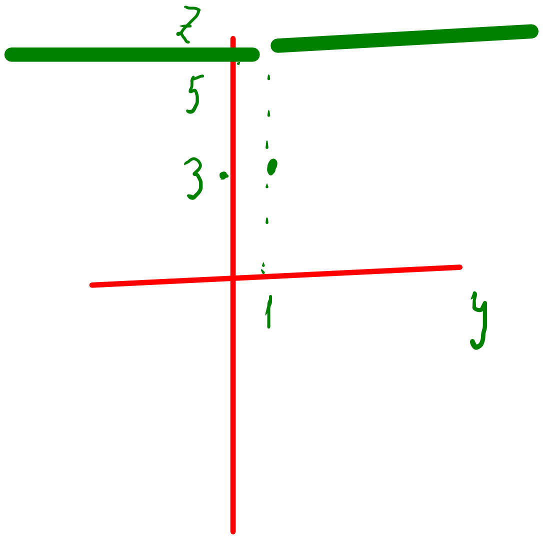
$$\lim_{x\to-\infty} y = \lim_{x\to-\infty} x^2 = +\infty$$

Perchè si mette l'ipotesi 2) nel teorema?

Es: $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = 1 \quad \forall x \in \mathbb{R}$

$\boxed{x_0 = 0}$

$$g(x) = \begin{cases} 3 & \text{se } y = 1 \\ 5 & \text{se } y \neq 1 \end{cases}$$



$g : \mathbb{R} \to \mathbb{R}$.

$(g \circ f)(x) = g(f(x)) = g(1) = 3 \quad \forall x \in \mathbb{R}$

$\to \lim_{x \to 0} (g \circ f)(x) = 3$

ma $\lim_{y\to y_0} g(y) = \lim_{y\to 1} g(y) = \underline{5}$

$y_0 = \lim_{x\to x_0} f(x) = \lim_{x\to 0} f(x) = 1$

$\Rightarrow \underset{3}{\lim_{x\to x_0} (g\circ f)(x)} \neq \underset{5}{\lim_{y\to y_0} g(y)}$

ma non vale l'ipotesi ?) .

e risponde (o 1) .

Esempio: (composizione con successioni)

- $b_n = \arctan(2n+1)$, $x_0 = +\infty$

$b = g \circ f$ con $g : \mathbb{R} = B \to \mathbb{R}$ $g(y) = \operatorname{artan}(y)$ $f : \mathbb{N} \to \mathbb{R}(=B)$

$f(n) =_{def} a_n = 2n+1 \geq n$

α) $\exists \lim\limits_{n \to +\infty (=x_0)} a_n = +\infty \, (=y_0)$ è di accumulazione per $B = \mathbb{R}$

β) $\forall n$ $a_n \neq y_0 \, (=+\infty)$ γ) $\exists \lim\limits_{y \to +\infty (=y_0)} \operatorname{artan}(y) = \frac{\pi}{2}$

sono quindi verificate le ipotesi del teorema

$$\exists \lim_{n \to x_0} \operatorname{artan}(2n+1) = \lim_{y \to y_0} \operatorname{artan}(y) = \frac{\pi}{2}$$

- - $b_n = e^{\frac{1}{n}}$, $n \in \mathbb{N}, n \geq 1$ $x_0 = +\infty$, $b = g \circ f$ $g : B = \mathbb{R} \to \mathbb{R}$ $g(y) = e^y$, $f : [1, +\infty) \cap \mathbb{N} \to B$ $f(n) =_{def} a_n$

α) $\exists \lim\limits_{n \to x_0} a_n = 0 \, (=y_0)$ β) g è continua in $y_0 \, (=0)$ γ) $\exists \lim\limits_{y \to 0} e^y = e^0 = 1$ quindi

$$\exists \lim_{n \to +\infty} b_n = \lim_{y \to 0} g(y) = 1$$

Prop: Se $\lim_{x \to x_0} f(x) = 0^+$ allora

$$\lim_{x \to x_0} \frac{1}{f(x)} = +\infty$$

($\exists\, \mathcal{U}$ intorno di x_0 : $f(x) \overset{>}{\neq} 0$ $x \in \mathcal{U} \setminus \{x_0\}$)
ma non per le ipotesi del Teo di comp.
ma per fare $1/f$

Se $\lim_{x \to x_0} f(x) = 0^-$ allora $\lim_{x \to x_0} \frac{1}{f(x)} = -\infty$

Se $\lim_{x \to x_0} f(x) = +\infty$ allora $\lim_{x \to x_0} \frac{1}{f(x)} = 0^+$

Se $\lim_{x \to x_0} f(x) = -\infty$ allora $\lim_{x \to x_0} \frac{1}{f(x)} = 0^-$

Se $\lim_{x \to x_0} f(x) = \ell$ con $\underline{\underline{\ell \neq 0}, +\infty, -\infty}$

allora $\lim_{x \to x_0} \frac{1}{f(x)} = \frac{1}{\ell}$.

quindi permanenza del segno $\exists\, \mathcal{U}$ int x_0
$\forall x \in \mathcal{U} \setminus \{x_0\}$
$f(x) \cdot \ell > 0$

$$f \to l \Rightarrow \frac{1}{f} \to \frac{1}{l}$$

// \\

$$\underbrace{\frac{1}{0^+} = +\infty \;,\; \frac{1}{0^-} = -\infty \;,\; \frac{1}{+\infty} = 0^+ \;,\; \frac{1}{-\infty} = 0^-}$$

$$f(x) \xrightarrow[x \to x_0]{} 0$$

$$x \xrightarrow[x \to 0]{} 0$$

$$\nexists \lim_{x \to 0} \frac{1}{x}$$

$$\lim_{\substack{x \to 0 \\ x > 0}} \frac{1}{x} = +\infty$$

$$\neq$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Prop: $a, b \in \overline{\mathbb{R}}$, $f : (a,b) \to \mathbb{R}$ con f debolmente crescente. Allora esistono

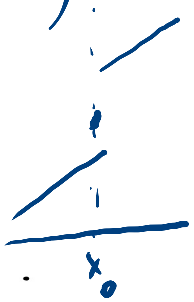
$$\boxed{\begin{gathered}\lim_{x\to a^+} f(x) = \inf_{x\in(a,b)} f(x) \\ \lim_{x\to b^-} f(x) = \sup_{x\in(a,b)} f(x)\end{gathered}}$$

Corollario

se f è monotona $f : A \to \mathbb{R}$

$\forall x_0 \in Acc(A)$

$\exists \lim_{x\to x_0^-} f(x)$ $\exists \lim_{x\to x_0^+} f(x)$

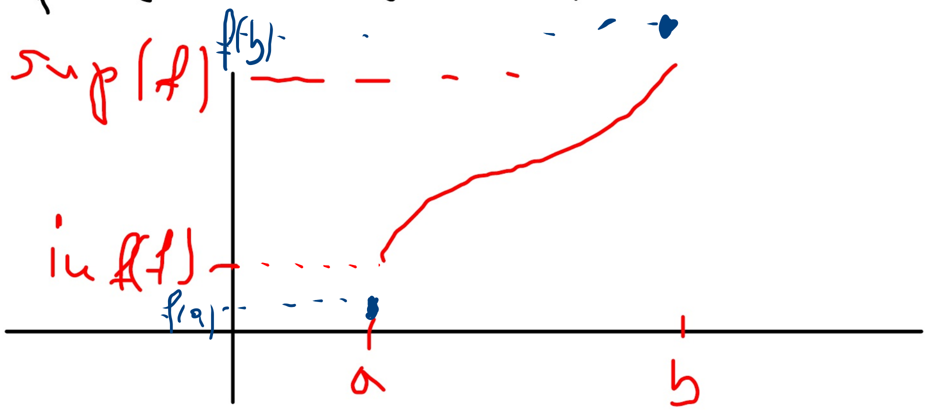


Analogo risultato se f è debolmente decrescente.

$f : [a;b] \to \mathbb{R}$ deb cres.

$f(a) = \min_{x\in[a,b]} f(x) \leq \inf_{x\in(a,b]} f = \lim_{a^+} f$

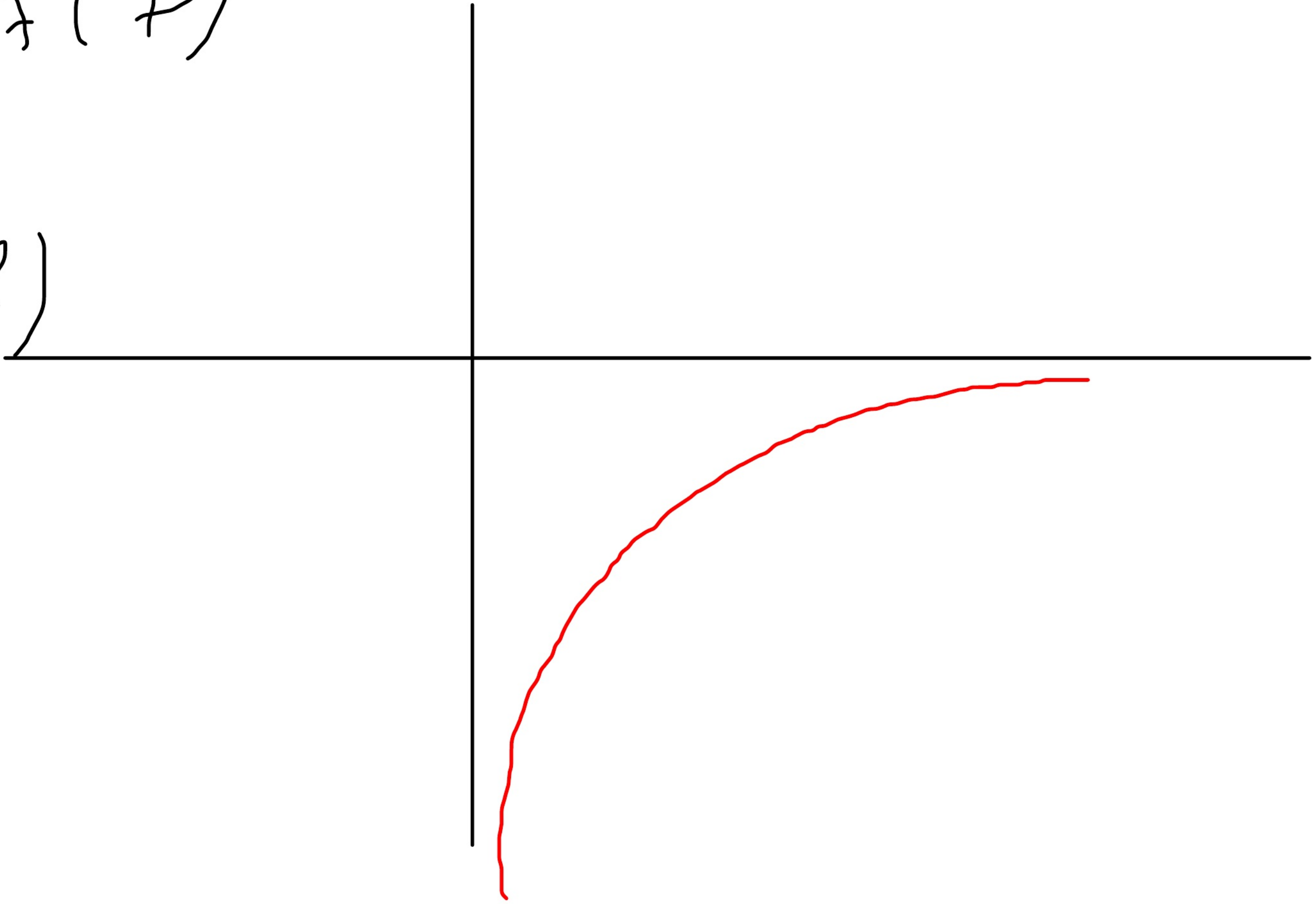
$f(b) = \max f \geq \sup_{[a,b)} f = \lim_{b^-} f$



Esempio: $f:(0,+\infty)\to\mathbb{R}$ $f(x)=-\frac{1}{x}$

$$\lim_{x\to 0^+} -\frac{1}{x} = -\infty = \inf(f)$$

$$\lim_{x\to +\infty} -\frac{1}{x} = 0 = \sup(f)$$



Limiti fondamentali

$$\lim_{x\to+\infty} x = +\infty$$

$x \in V$

$\forall M^*$ $\quad \exists\, \mathcal{U}(+\infty)$: $x \in \mathcal{U} \Rightarrow x \geq M$

$x \geq M$

$V(\infty) = [M, +\infty)$ $\qquad \mathcal{U} \overset{\shortparallel}{} \; V = [M, +\infty)$

$n \in \mathbb{N}$ $\qquad x^n = x \cdot x \cdot \ldots \cdot x$ $\quad$ n volte

$*\; M \geq 1$ $\quad x^n \geq x$

$x \geq M \geq 1$

$$\Rightarrow \lim_{x\to+\infty} x^n = \left(\lim_{n\to\infty} x\right)\cdot\left(\lim_{n\to\infty} x\right)\cdots = (+\infty)(+\infty)\cdots$$

$$= +\infty$$

$$\lim_{x\to+\infty} \frac{1}{x} = \frac{1}{+\infty} = 0 \;, \qquad \lim_{x\to+\infty} \frac{1}{x^n} = 0$$

limiti di polinomi.

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 \cdot x + a_0$$

$a_0, a_1, \dots, a_n \in \mathbb{R}$, n è il grado del polinomio

$n \in \mathbb{N}$.

$$\lim_{x \to +\infty} p(x) = ?$$

Es: $$\lim_{x \to +\infty} 3x^2 - 7x + 1 = +\infty - \infty + 1 = ?$$

$n > m$

$x^n > x^m$

$x \to +\infty$

si può assumere

$x > \underline{1}$

$$\lim_{x\to\infty} 3x^2\left(1-\frac{7x}{3x^2}+\frac{1}{3x^2}\right)=$$

$$=\lim_{x\to\infty} \boxed{3}x^2\left(1-\frac{7}{3x}+\frac{1}{3x^2}\right)=+\infty\left(1-\frac{7}{+\infty}+\frac{1}{+\infty}\right)=$$

$$=+\infty(1-0-0)=+\infty.$$

$$p(x)=a_n x^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0=$$

$$=a_n x^n\left(1+\frac{a_{n-1}}{a_n}\cdot\underset{\longrightarrow 0}{\boxed{\frac{x^{n-1}}{x^n}}}+\dots+\frac{a_1}{a_n}\underset{\longrightarrow 0}{\boxed{\frac{x}{x^n}}}+\frac{a_0}{a_n}\underset{\longrightarrow 0}{\boxed{\frac{1}{x^n}}}\right)$$

$n>m \Rightarrow x^n>x^m$ $\qquad n-m>0$

$x>1$

$$\frac{x^m}{x^n}=\frac{1}{x^{n-m}}\longrightarrow 0$$

se $x\longrightarrow+\infty$

o anche se $x\longrightarrow-\infty$

$$\lim_{x\to+\infty} \boxed{a_n x^n} + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = \lim_{x\to+\infty} a_n x^n$$

$$\lim_{x\to-\infty} a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = \lim_{x\to-\infty} a_n x^n$$

Es: $$\lim_{x\to-\infty} \boxed{-2x^5} + 3x^2 = \lim_{x\to-\infty} -2x^5 = -2(-\infty)^5 =$$

$$= (-2)(-\infty) = +\infty.$$

Funzioni razionali

$\frac{p(x)}{q(x)}$ $\quad p, q$ polinomi.

$$p(x) = a_n x^n + \dots + a_1 x + a_0$$

$$q(x) = b_m x^m + \dots + b_1 x + b_0$$

$$\lim_{x \to \pm\infty} \frac{p(x)}{q(x)} = \lim_{x \to \pm\infty} \frac{a_n x^n \left(1 + \frac{a_{n-1}}{a_n} \overbrace{\frac{x^{n-1}}{x^n}}^{\to 0} + \dots + \frac{a_0}{a_n} \overbrace{\frac{1}{x^n}}^{\to 0}\right)}{b_m x^m \left(1 + \frac{b_{m-1}}{b_m} \underbrace{\frac{x^{m-1}}{x^m}}_{\to 0} + \dots + \frac{b_0}{b_m} \underbrace{\frac{1}{x^m}}_{\to 0}\right)}$$

$$= \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m} \lim_{x \to \pm\infty} \frac{x^n}{x^m}$$

Es:

$$\lim_{x\to+\infty}\frac{7x^4+5x^2}{-2x^3+x}=$$

$$=\lim_{x\to+\infty}\frac{7x^4}{-2x^3}=\lim_{x\to+\infty}\frac{7x}{-2}=-\infty$$

Rapporto di polinomi senza termine noto per $x\to0^+$: forma ind. $\frac{0}{0}$

si raccoglie il monomio di grado **minimo**

cioè il monomio in modulo più grosso per $x\to0^+$

$$\lim_{x\to0^+}\frac{x^3-2x^2+x}{x^2-x}=\lim_{x\to0^+}\frac{(x^2-2x+1)}{x-1}\frac{x}{x}=$$

$$=\lim_{x\to0^+}\frac{x^2-2x+1}{x-1}=-1\cdot1=-1$$

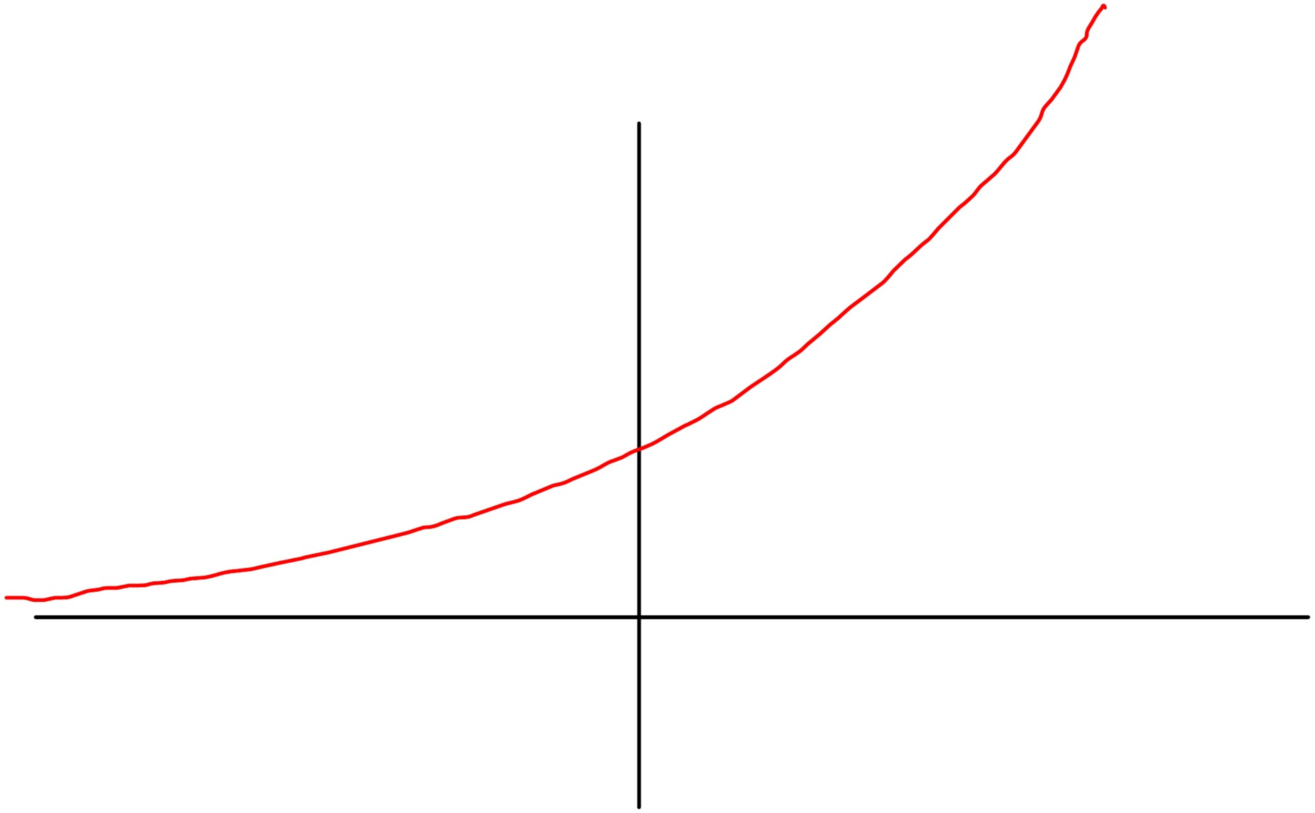
$$\lim_{x\to0^+}\frac{1}{x-1}=-1$$

$$\lim_{x\to0^+}x^2-2x+1=1$$

Limiti fondamentali

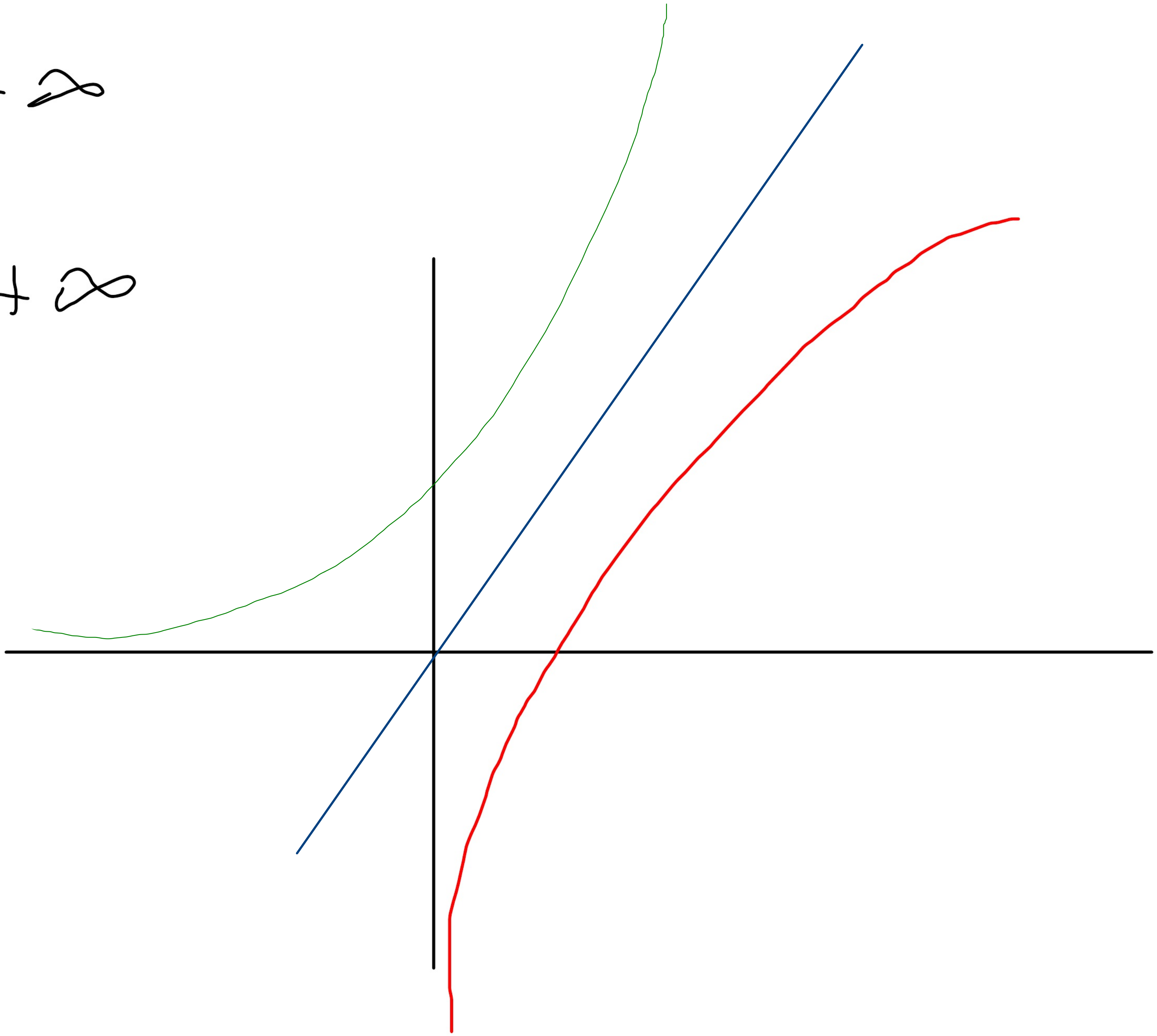
$$\lim_{x \to +\infty} e^x = +\infty$$

$$\lim_{x \to -\infty} e^x = 0^+$$



$$\lim_{x\to 0^+} \log x = -\infty$$

$$\lim_{x\to +\infty} \log x = +\infty$$



Limiti notevoli

$$\lim_{x\to 0} \cos x = 1$$

$$\lim_{x\to 0} \sin x = 0$$

*

$$0 \le |\sin x| \le |x|$$

$$\downarrow \qquad \downarrow \qquad \downarrow_{x\to 0}$$

$$0 \qquad 0 \qquad 0$$

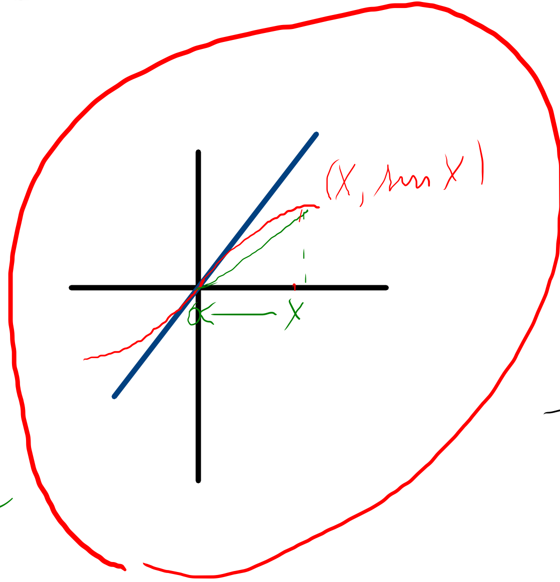
$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

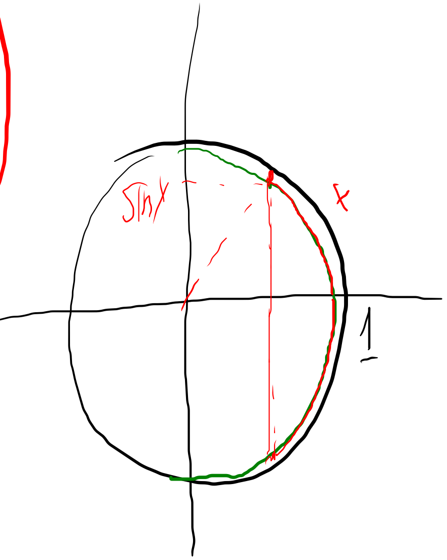
è indeterminata perché $\lim_{x\to 0} \sin x = 0$

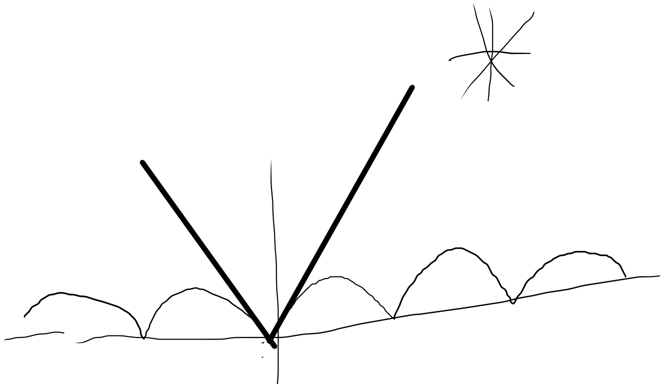
$\lim_{x\to 0} x = 0$.

$$\frac{\sin x}{x} \xrightarrow[x\to 0]{} 1$$

$\uparrow$ coeff. della corda





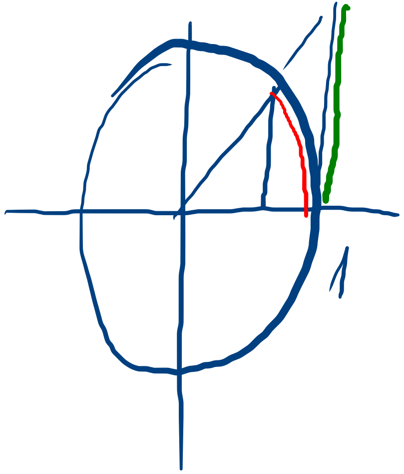


Se $x \to 0$ posso pensare $|x| < \frac{\pi}{2}$

$\underline{\underline{|x|<\frac{\pi}{2}}}$

$\frac{\sin x}{x} \leq 1$

$\frac{\operatorname{tg} x}{x} \geq 1$



x l'arco

$\sin x$ corda

$\operatorname{tg} x$ tangente

$$\begin{tikzcd} 1 \arrow[d] & \geq \frac{\sin x}{x} \arrow[d] & \geq \cos x \arrow[r, "x \to 0"'] & 1 \\ 1 & 1 & & \end{tikzcd}$$

carabinieri

$\sin x \xrightarrow[x \to 0]{} 0$

$$\begin{tikzcd} \cos^2 x \arrow[d] & + \sin^2 x \arrow[d] & = 1 \arrow[d] \\ 1 & 0 & 1 \end{tikzcd}$$

ma $\cos x > 0$ $|x| < \frac{\pi}{2}$

$\Downarrow$

$\cos x \to 1$

$\frac{\operatorname{tg} x}{x} \xrightarrow[x \to 0]{} 1$

$$\boxed{\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}}$$

$$\frac{1-\cos x}{\frac{x^2}{2}} \xrightarrow[x\to 0]{} 1$$

dim:

$$\frac{1-\cos x}{x^2} = \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} =$$

$$= \frac{1-\cos^2 x}{x^2(1+\cos x)} = \frac{\sin^2 x}{x^2(1+\cos x)} = \underset{\substack{\downarrow\\ 1}}{\boxed{\frac{\sin x}{x}}} \cdot \underset{\substack{\downarrow\\ 1}}{\boxed{\frac{\sin x}{x}}} \cdot \underset{\substack{\downarrow\\ \frac{1}{1+1}}}{\boxed{\frac{1}{1+\cos x}}}$$

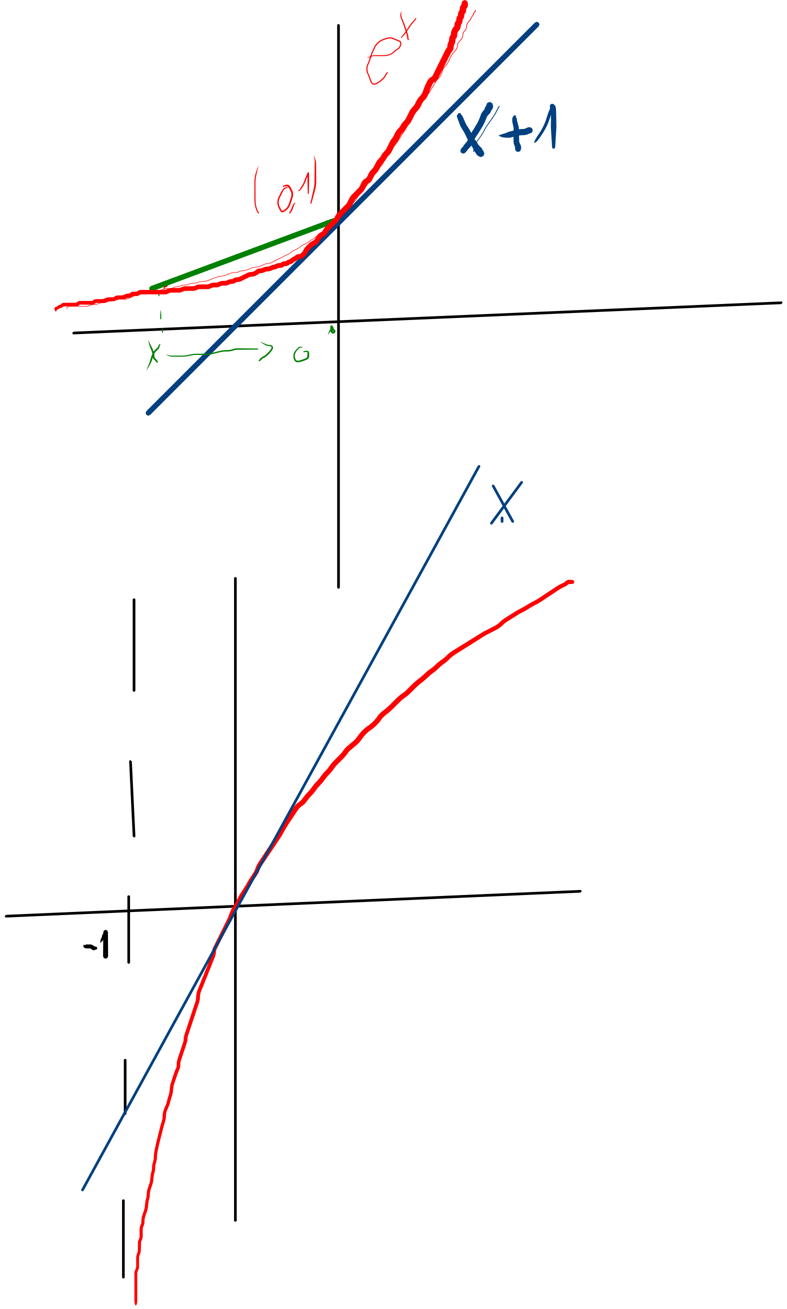
$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.$$

e^0

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

pendenze corde

$$\lim_{x\to 0}\frac{\log(1+x)}{x}=1.$$



Limiti fondamentali

$$\lim_{x \to +\infty} a^x = \begin{cases} +\infty & \text{se } a>1 \\ 1 & \text{se } a=1 \\ 0^+ & \text{se } 0<a<1 \end{cases}$$

$0 < a < 1$

a^x $\quad x \to +\infty$

$b = \frac{1}{a} > 1$

$$\left(\frac{1}{b}\right)^x = \frac{1}{b^x} \to 0, \qquad b^x \to +\infty$$

$-x=y$

$$\lim_{x \to -\infty} a^x = \lim_{y \to +\infty} a^{-y} = \lim_{y \to +\infty} \frac{1}{a^y}$$

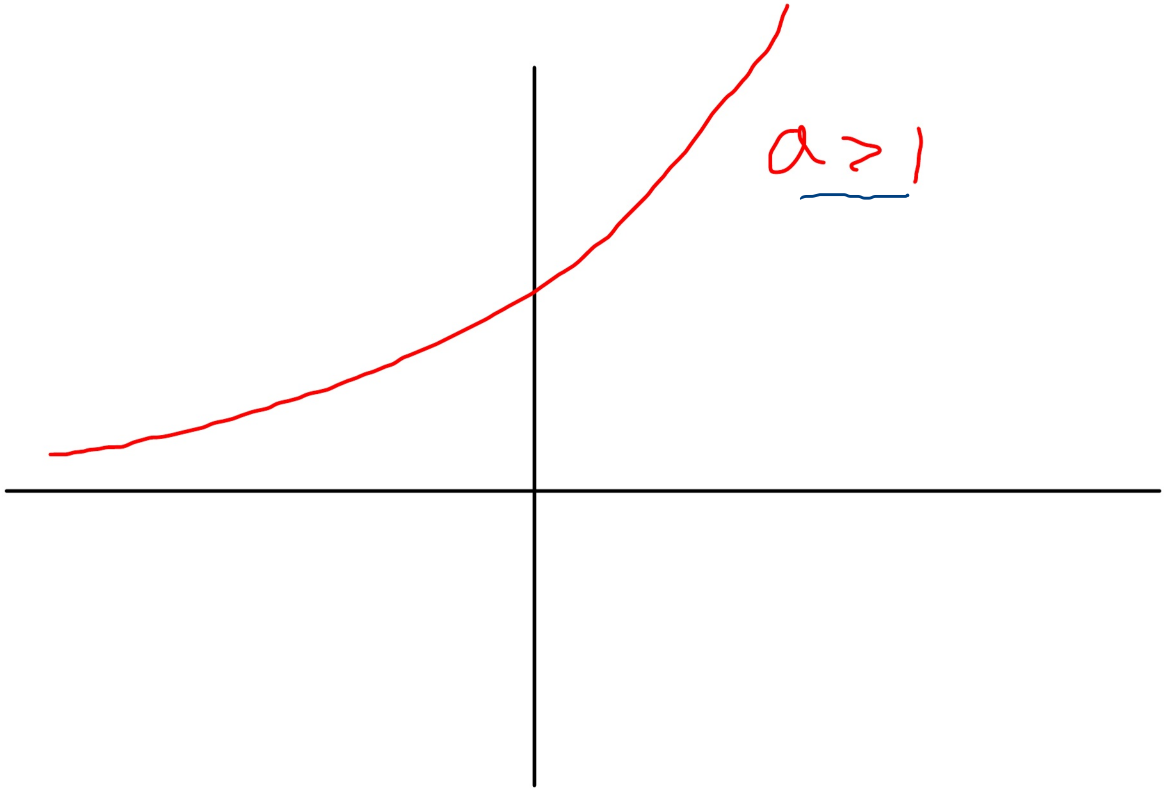
cambio variabile $y = -x$ quindi se $x \to -\infty$

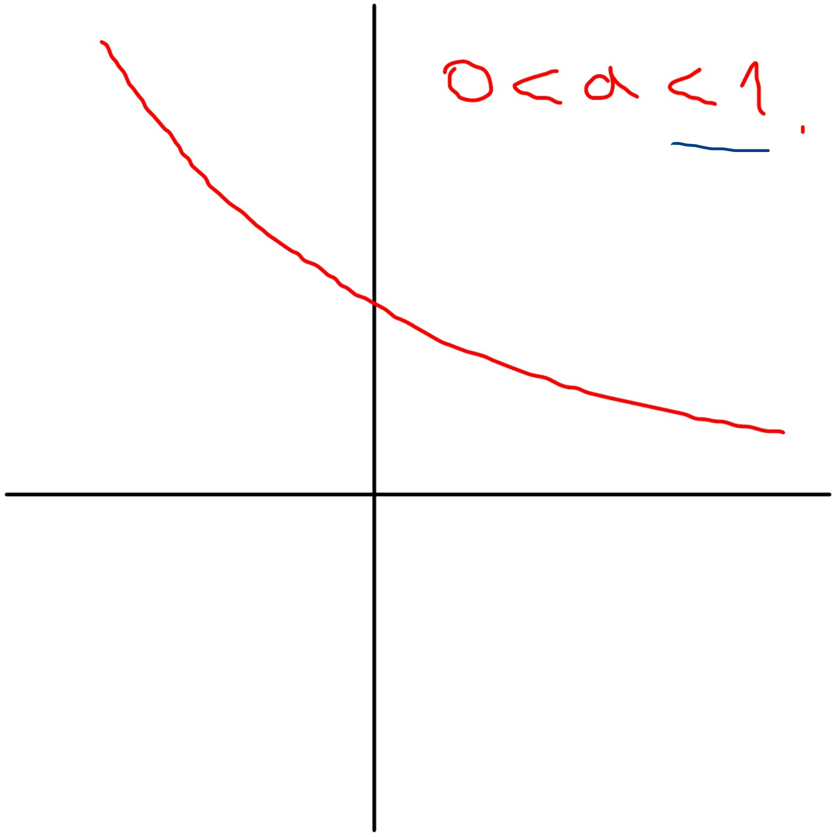
allora $y \to +\infty$

$a>1$, $e^{\lg a}$

$$a^x = e^{x \cdot \boxed{\lg a}}, \quad \lg a > 0$$

$$\Rightarrow \lim_{x\to-\infty} a^{x} = \begin{cases} 0^{+} & \text{se } a>1 \\ 1 & \text{se } a=1 \\ +\infty & \text{se } 0<a<1 \end{cases}$$





"Potenze" $\alpha \in \mathbb{R}$ $\underline{x^{\alpha}}$

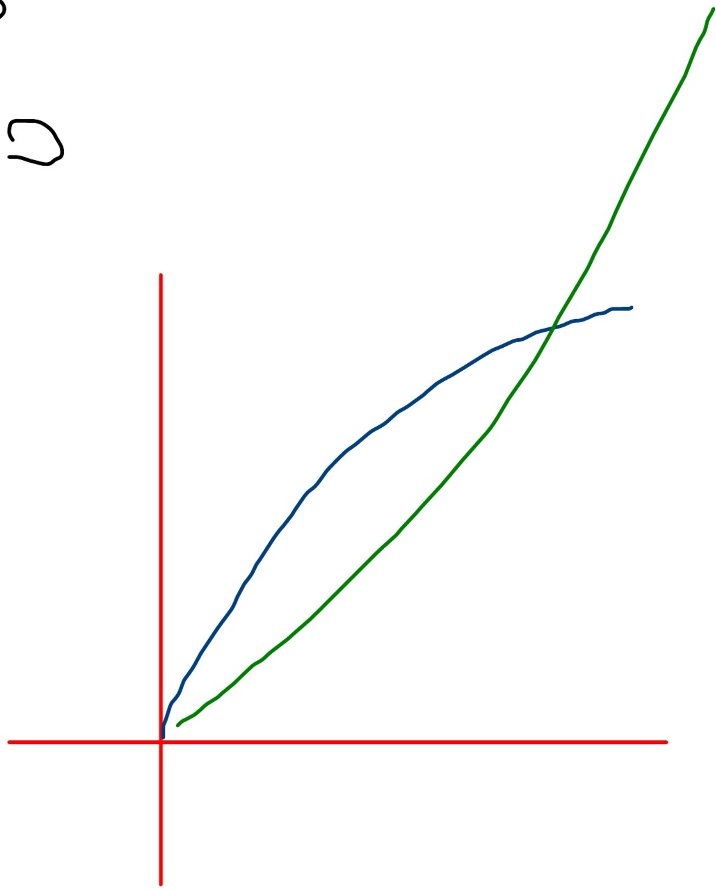
$x \to +\infty$

$x > 0$, $x > 1$

$x > 0$

$$\lim_{x \to +\infty} x^{\alpha} = \begin{cases} +\infty & \text{se } \alpha > 0 \\ 1 & \text{se } \alpha = 0 \\ 0^{+} & \text{se } \alpha < 0 \end{cases}$$

$\leftarrow$ $\alpha > 0$ $\underline{x^{\alpha} \geq x^{[\alpha]}}$



Confronti tra infiniti

$$\lim_{x \to +\infty} \frac{a^x}{x^\alpha} = \begin{cases} +\infty & \text{se } a > 1 \\ 0^+ & \text{se } 0 < a < 1. \end{cases}$$

caso interessat
$\alpha > 0$
$\frac{\infty}{\infty}$

$a^x \xrightarrow[x \to +\infty]{} 0$

caso interessante è $\alpha < 0$ $\quad x^\alpha \xrightarrow[x \to +\infty]{} 0$ $\quad \frac{0}{0}$

se $a = 1 \Rightarrow a^x = 1$

$$\Rightarrow \lim_{x \to +\infty} \frac{a^x}{x^\alpha} = \lim_{x \to \infty} \frac{1}{x^\alpha}$$

caso precedente.

Es: $a = \frac{1}{2}$ $\alpha = -3$

$$\lim_{x \to +\infty} \frac{a^x}{x^\alpha} = \lim_{x \to +\infty} \frac{\left(\frac{1}{2}\right)^x}{x^{-3}} = \lim_{x \to +\infty} \boxed{\frac{x^3}{2^x}} = 0^+$$

$$\lim_{x \to +\infty} \frac{2^x}{x^3} = +\infty \nearrow$$

Confronto fra logaritmo e potenze

$$\lim_{x\to+\infty} \frac{\log x}{x}$$

cambio di variabile

$y = \log x \Rightarrow x = e^y$

se $x \to +\infty \Rightarrow y = \log x \to +\infty$

$$\left(\lim_{x\to+\infty} \log x = +\infty\right)$$

$$\lim_{x\to+\infty} \frac{\log x}{x} = \lim_{y\to+\infty} \frac{y}{e^y} = 0$$

$$\lim_{x\to+\infty}\frac{(\log x)^{\beta}}{x^{\alpha}} = \circledast \qquad \alpha,\beta\in\mathbb{R},\ \underline{\underline{\alpha,\beta>0}}.$$

cambio di variabile $y = \lg x \Rightarrow x = e^{y}$

se $x \to +\infty \Rightarrow y \to +\infty$

$$\circledast = \lim_{y\to\infty}\frac{y^{\beta}}{(e^{y})^{\alpha}} = \lim_{y\to\infty}\frac{y^{\beta}}{e^{y\cdot\alpha}} = \lim_{y\to\infty}\frac{y^{\beta}}{(\underline{e^{\alpha}})^{y}} =$$

$$= \lim_{y\to\infty}\frac{y^{\beta}}{a^{y}} = 0 \quad \boxed{\begin{array}{c}\text{con } a = e^{\alpha} \text{ ma } \alpha>0 \\ \Rightarrow \underline{a = e^{\alpha} > 1}\end{array}}$$

$$\boxed{\lim_{x\to 0^+} x\log x = 0}\,(-\infty) = \text{indeterminata.}$$

cambio di variabile $\underline{y=\log x}$, $\underline{x=e^{y}}$

se $x\to 0^+ \Rightarrow y=\log x \longrightarrow -\infty$

$$\lim_{y\to-\infty} e^{y}\cdot y = e^{-\infty}\cdot(-\infty) = 0^+(-\infty)$$

indeterminata

$$\uparrow$$

$$\lim_{y\to-\infty} e^{y}$$

$z=-y$ se $y\longrightarrow -\infty \Rightarrow z \longrightarrow +\infty$

Altro modo $\dfrac{\lg x}{1/x} = -\dfrac{\lg 1/x}{1/x} \overset{x\to 0^+}{=} -\dfrac{\lg y}{y}$ $y\longrightarrow +\infty$

$$\lim_{y\to-\infty} e^{y}\cdot y = \lim_{z\to+\infty} e^{-z}(-z) =$$

$$= \lim_{z\to+\infty} \frac{-z}{e^{z}} = 0$$

$$\boxed{\lim_{x\to 0^+} x\log x = 0}$$

Osservazione: per il teorema del limite di composizione

$$\boxed{0} = \lim_{n\to+\infty} \frac{1}{n}\log\frac{1}{n} = \lim_{n\to+\infty} -\frac{1}{n}\log n = \boxed{-\lim_{n\to+\infty} \frac{1}{n}\log n}$$

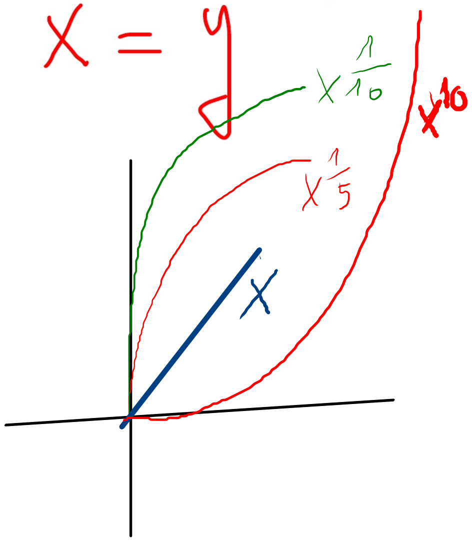
$$\boxed{\alpha > 0}$$

$$\lim_{x \to 0^+} x^{\alpha} \log x$$

sostituisco $y = x^{\alpha}$

quindi $x = y^{1/\alpha}$

se $x \to 0 \Rightarrow y = x^{\alpha} \to 0$



$$= \lim_{y \to 0^+} y \cdot \log\left(y^{1/\alpha}\right) =$$

$$= \lim_{y \to 0^+} y \cdot \frac{1}{\alpha} \log y = \frac{1}{\alpha} \cdot \lim_{y \to 0^+} y \log y = 0$$

$$(1+x)^{1/x} = e^{\underset{+\infty}{\underset{\downarrow}{\frac{1}{x}}}\,\underset{0}{\underset{\downarrow\, x\to 0^+}{\log(1+x)}}} \longrightarrow e^{1}$$

Esercizio :

$$\boxed{\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e} = (1+0)^{\frac{1}{0^+}} = 1^{\infty} \;?$$

$$(1+x)^{\frac{1}{x}} = e^{\log\left((1+x)^{\frac{1}{x}}\right)} = e^{\underbrace{\frac{1}{x}\log(1+x)}_{y}}$$

sostituzione $y = \frac{1}{x}\log(1+x)$

se $x \to 0^+$ $y \to$? $\lim_{x\to 0^+} \frac{1}{x}\log(1+x) = 1$

limite notevole visto prima.

$$\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = \lim_{y\to 1} e^{y} = e^{1} = e.$$

$$\boxed{\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = e}$$

Osservazione: $b_n = \left(1+\frac{1}{n}\right)^n$ $\quad n = \frac{1}{\frac{1}{n}}$ $\quad b = g\circ f$ $\quad g(y) = (1+y)^{1/y}$ $\quad y>0$

$f(n) =: a_n = \frac{1}{n}$

$x_0 = +\infty$, $a_n \to 0^+ = y_0$ è di accumulazione per dom g, $a_n = \frac{1}{n} \neq y_0 = 0^+$

quindi $\boxed{\lim_{\substack{n\to+\infty \\ (x_0)}} \underline{\left(1+\frac{1}{n}\right)}^{n}} = \lim_{y\to 0^+}(1+y)^{1/y} = \boxed{e}$

Nuovi casi di indeterminazione

$f(x) > 0$

$$\lim_{x \to x_0} f(x)^{g(x)}$$

quando è una forma indeterminata?

$$\boxed{f(x)^{g(x)}} = e^{\operatorname{Log}\left(f(x)^{g(x)}\right)} = e^{\underline{\boxed{g(x)\log(f(x))}}}$$

quando è indeterminato il limite

$$\lim_{x \to x_0} g(x) \cdot \log(f(x)) \quad ?$$

1) $\boxed{g \to 0}$, $\boxed{f \to +\infty}$ $\to$ $\log(f) \to +\infty$

$0 \cdot \infty$ quindi $(+\infty)^0$ è indeterminata

2) $g \to 0 \quad , \quad f \to 0^+ \Rightarrow \log f \to -\infty$

$$\Rightarrow g \cdot \log(f) = 0 \cdot (-\infty) = ?$$

$$\left(\underset{\underset{f}{\uparrow}}{0^+}\right)^{0 \leftarrow g}$$

3) $g \longrightarrow \pm\infty$, $f \longrightarrow 1$ quidi $\log(f) \longrightarrow 0$

$g \cdot \log(f) = \pm\infty \cdot 0 = ?$

$(1)^{\pm\infty}$ è indeterminata

$(+\infty)^0$, $(0^+)^0$, $(1)^{+\infty}$, $(1)^{-\infty}$

sono forme indeterminate

Es: $\lim_{x\to 0^+} x^x =$

$$= \lim_{x\to 0^+} e^{\log(x^x)} = \lim_{x\to 0^+} e^{\overset{\xrightarrow{x\to 0^+} 0}{\boxed{x\log x}}} = e^0 = 1$$

esponenziale è continua

$y = x\log x$ se $x \to 0^+ \Rightarrow y \to 0$

$$\lim_{y\to 0} e^y = e^0 = 1.$$

$\forall f \; \exists \lim_{x_0} f \in \mathbb{R}$ $\quad \lim_{x \to x_0} g(f(x)) = g\left(\lim_{x \to x_0} f(x)\right)$ $\quad \mathcal{H}$

Teo di cambiamento di variabile limite $\Uparrow$ $\quad \Downarrow$?

Riflessione per deciderne la verità... ?

$\forall f \; \exists \lim_{x \to x_0} f \in \mathbb{R}$ se g è continua in $\lim_{x \to x_0} f(x)$ $\quad Th$

A Ricevimento

1) $\lim_{x_0} f \in \operatorname{dom} g$ ma è isolato ? ok

2) $\lim_{x_0} f \in \operatorname{dom} g \cap \operatorname{Acc}(\operatorname{dom} g)$ $\quad$? $\forall \varepsilon \; \exists \delta \; \forall y : |y - y_0| < \delta \Rightarrow |g(y) - g(y_0)| < \varepsilon$?

$\lim_{x_0} f = y_0$

$x_0 = +\infty$

Per $\mathcal{H}$: $f(n) = y_n \longrightarrow y_0 \Rightarrow g(y_n) \longrightarrow g(y_0)$

non Th $\quad \exists \varepsilon_0 \; \forall \delta > 0 \; \exists y_\delta \; |y - y_0| < \delta$ e $|g(y) - g(y_0)| \geq \varepsilon_0$

$\delta = \frac{1}{n}$ $\quad \exists y_n \; |y_n - y_0| < \frac{1}{n}$ e $|g(y_n) - g(y_0)| \geq \varepsilon_0$