Prop: Se $\lim_{x \to x_0} f(x) = 0^+$ allora

$$\lim_{x \to x_0} \frac{1}{f(x)} = +\infty$$

Se $\lim_{x \to x_0} f(x) = 0^-$ allora $\lim_{x \to x_0} \frac{1}{f(x)} = -\infty$

Se $\lim_{x \to x_0} f(x) = +\infty$ allora $\lim_{x \to x_0} \frac{1}{f(x)} = 0^+$

Se $\lim_{x \to x_0} f(x) = -\infty$ allora $\lim_{x \to x_0} \frac{1}{f(x)} = 0^-$

Se $\lim_{x \to x_0} f(x) = l$ con $l \neq 0, +\infty, -\infty$

allora $\lim_{x \to x_0} \frac{1}{f(x)} = \frac{1}{l}$.

$$f \to \ell \Rightarrow \frac{1}{f} \to \frac{1}{\ell}$$

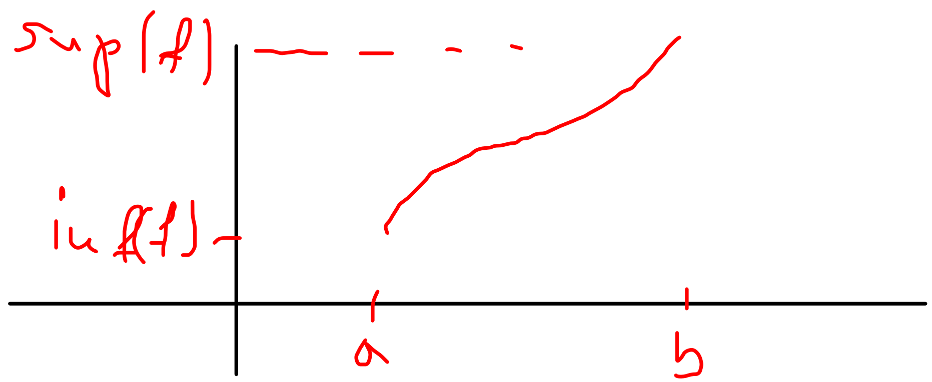
$$\frac{1}{0^+} = +\infty, \quad \frac{1}{0^-} = -\infty, \quad \frac{1}{+\infty} = 0^+, \quad \frac{1}{-\infty} = 0^-$$

Prop: $a, b \in \overline{\mathbb{R}}$, $f : (a,b) \to \mathbb{R}$ con f debolmente crescente. Allora esistono

$$\lim_{x \to a^+} f(x) = \inf_{x \in (a,b)} f(x)$$

$$\lim_{x \to b^-} f(x) = \sup_{x \in (a,b)} f(x)$$

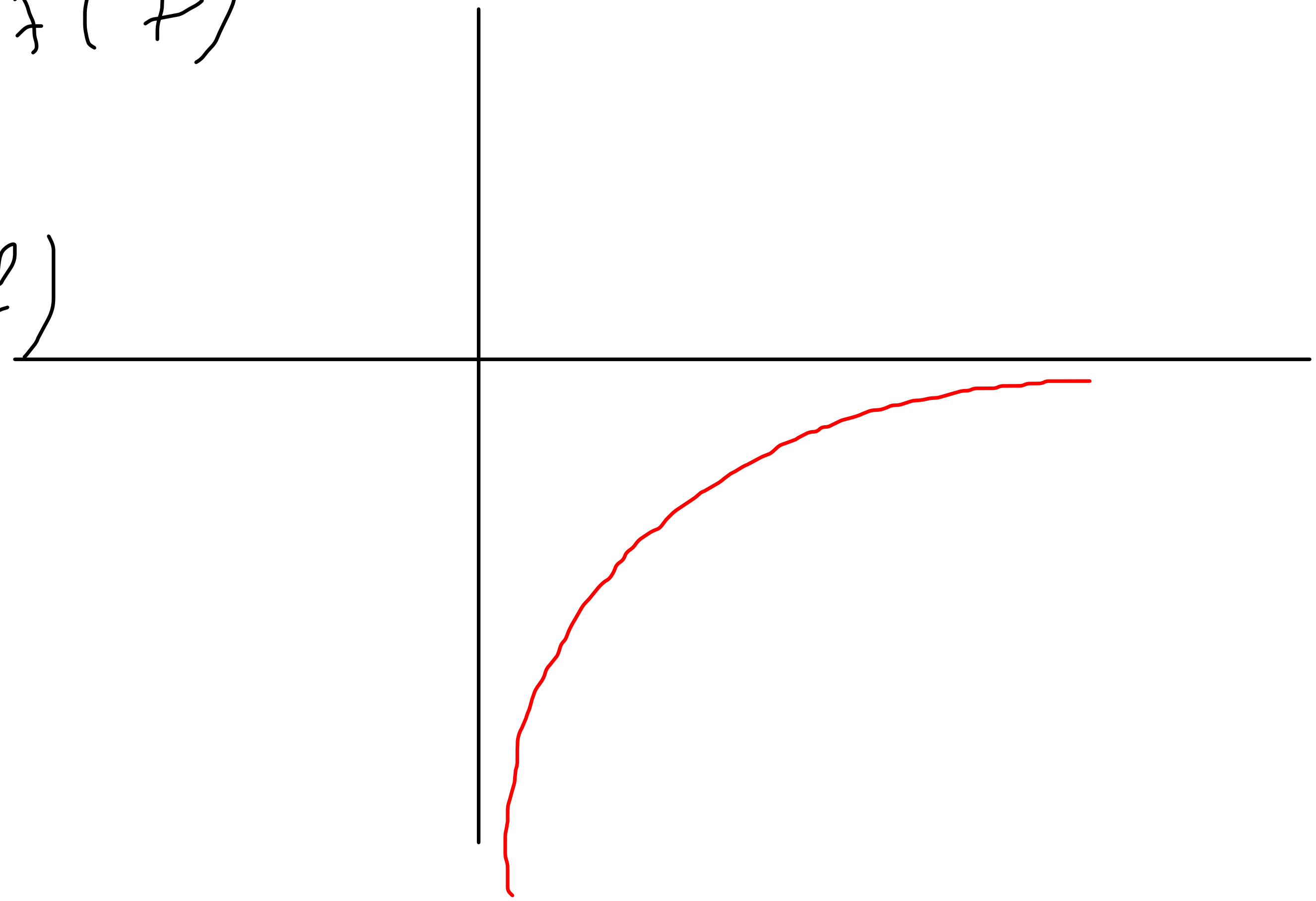
Analogo risultato se f è debolmente decrescente.



Esempio: $f : (0, +\infty) \to \mathbb{R}$ $\quad f(x) = -\frac{1}{x}$

$$\lim_{x \to 0^+} -\frac{1}{x} = -\infty = \inf(f)$$

$$\lim_{x \to +\infty} -\frac{1}{x} = 0 = \sup(f)$$



Limiti fondamentali

$$\lim_{x \to +\infty} x = +\infty$$

$n \in \mathbb{N}$ $\quad x^n = x \cdot x \cdot \ldots \cdot x$ $\quad$ n. volte

$$\Rightarrow \lim_{x \to +\infty} x^n = \left(\lim_{n \to \infty} x\right) \cdot \left(\lim_{n \to \infty} x\right) \ldots = (+\infty)(+\infty) \ldots$$

$$= +\infty$$

$$\lim_{x \to +\infty} \frac{1}{x} = \frac{1}{+\infty} = 0 \quad , \quad \lim_{x \to +\infty} \frac{1}{x^n} = 0$$

Limiti di polinomi.

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 \cdot x + a_0$$

$a_0, a_1, \dots, a_n \in \mathbb{R}$, n è il grado del polinomio

$n \in \mathbb{N}$.

$$\lim_{x \to +\infty} p(x) = ?$$

Es: $\lim_{x \to +\infty} 3x^2 - 7x + 1 = +\infty - \infty + 1 = ?$

$$\lim_{x\to\infty} 3x^2\left(1-\frac{7x}{3x^2}+\frac{1}{3x^2}\right)=$$

$$=\lim_{x\to\infty} 3x^2\left(1-\frac{7}{3x}+\frac{1}{3x^2}\right)=+\infty\left(1-\frac{7}{+\infty}+\frac{1}{+\infty}\right)=$$

$$=+\infty(1-0-0)=+\infty.$$

$$p(x)=a_n x^n+a_{n-1}x^{n-1}+\dots+a_1 x+a_0=$$

$$=a_n x^n\left(1+\frac{a_{n-1}}{a_n}\cdot\frac{x^{n-1}}{x^n}+\dots+\frac{a_1}{a_n}\frac{x}{x^n}+\frac{a_0}{a_n}\frac{1}{x^n}\right)$$

$\frac{x^{n-1}}{x^n}\to 0$ $\quad\frac{x}{x^n}\to 0$ $\quad\frac{1}{x^n}\to 0$

se $x\to+\infty$

o anche se $x\to-\infty$

$$\lim_{x\to+\infty} a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = \lim_{x\to+\infty} a_n x^n$$

$$\lim_{x\to-\infty} a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = \lim_{x\to-\infty} a_n x^n$$

Es: $\lim_{x\to-\infty} -2x^5 + 3x^2 = \lim_{x\to-\infty} -2x^5 = -2(-\infty)^5 =$

$= (-2)(-\infty) = +\infty.$

Funzioni razionali

$$\frac{p(x)}{q(x)} \qquad p, q \text{ polinomi.}$$

$$p(x) = a_n x^n + \dots + a_1 x + a_0$$

$$q(x) = b_m x^m + \dots + b_1 x + b_0$$

$$\lim_{x \to \pm\infty} \frac{p(x)}{q(x)} = \lim_{x \to \pm\infty} \frac{a_n x^n \left(1 + \frac{a_{n-1}}{a_n} \frac{x^{n-1}}{x^n} + \dots + \frac{a_0}{a_n} \frac{1}{x^n}\right)}{b_m x^m \left(1 + \frac{b_{m-1}}{b_m} \frac{x^{m-1}}{x^m} + \dots + \frac{b_0}{b_m} \frac{1}{x^m}\right)}$$

$$\to 0 \qquad \to 0 \qquad \to 0 \qquad \to 0$$

$$= \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}$$

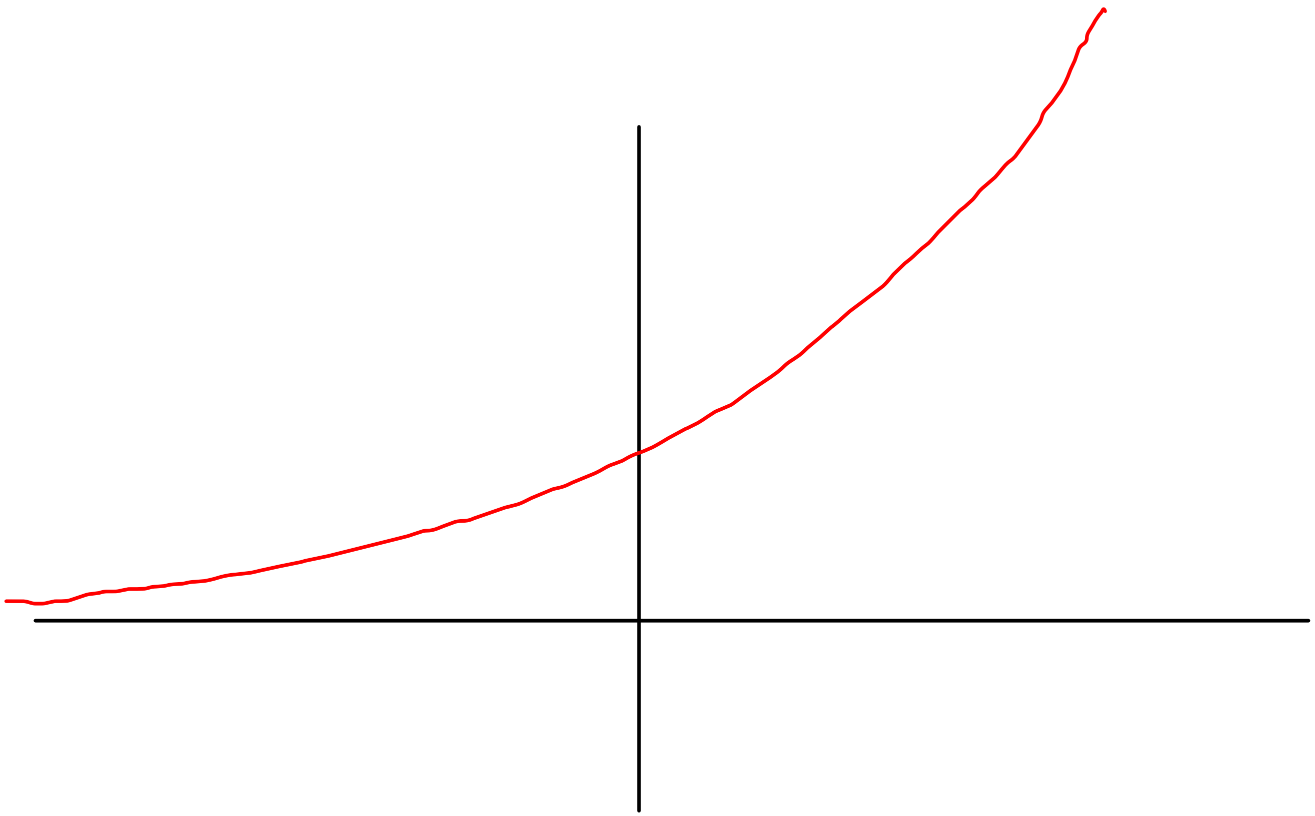
Es:
$$\lim_{x \to +\infty} \frac{7x^4 + 5x^2}{-2x^3 + x} =$$

$$= \lim_{x \to +\infty} \frac{7x^4}{-2x^3} = \lim_{x \to +\infty} \frac{7x}{-2} = -\infty$$

Limiti fondamentali

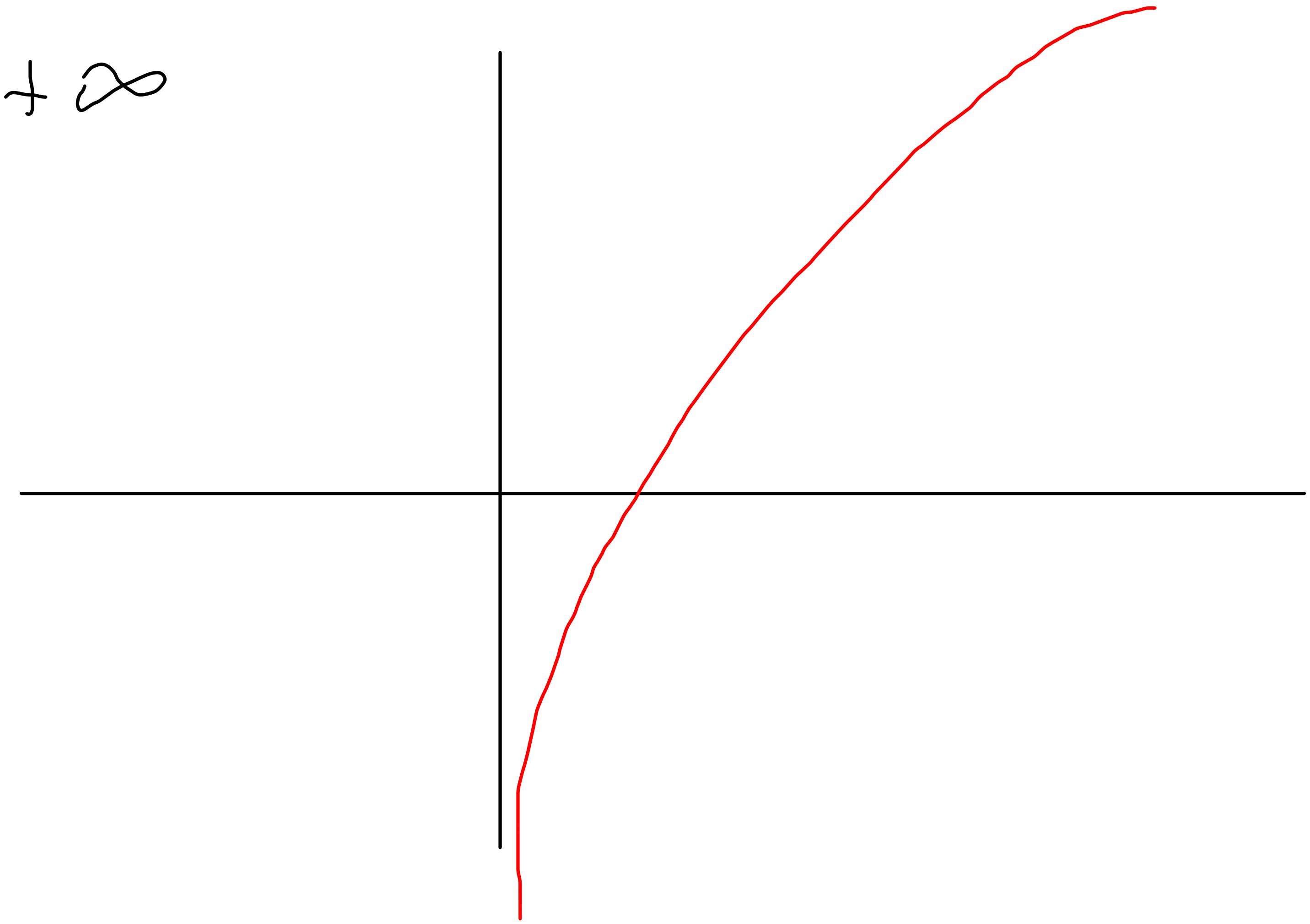
$$\lim_{x \to +\infty} e^x = +\infty$$

$$\lim_{x \to -\infty} e^x = 0^+$$



$$\lim_{x \to 0^+} \log x = -\infty$$

$$\lim_{x \to +\infty} \log x = +\infty$$



Limiti notevoli

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

è indeterminata perché $\lim_{x\to 0} \sin x = 0$

$\lim_{x\to 0} x = 0$.

$$\boxed{\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}}$$

dim:

$$\frac{1-\cos x}{x^2}=\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}=$$

$$=\frac{1-\cos^2 x}{x^2(1+\cos x)}=\frac{\sin^2 x}{x^2(1+\cos x)}=\underbrace{\boxed{\frac{\sin x}{x}}}_{\downarrow\ 1}\cdot\underbrace{\boxed{\frac{\sin x}{x}}}_{\downarrow\ 1}\cdot\underbrace{\boxed{\frac{1}{1+\cos x}}}_{\downarrow\ \frac{1}{1+1}}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}.$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1.$$

Limite della composizione di funzioni

Teorema: $A, B \subset \mathbb{R}$, $f : A \to B$, $g : B \to \mathbb{R}$, $x_0 \in Acc(A)$. Se esiste $\lim_{x \to x_0} f(x) = y_0$

e $y_0 \in Acc(B)$ e $\exists \lim_{y \to y_0} g(y) = \ell \in \overline{\mathbb{R}}$

e se è verificato almeno una delle 2 seguenti ipotesi:

1) $y_0 \in B$ e g è continua in y_0

2) esiste U intorno di x_0 t.c.

se $x \in U \cap A \setminus \{x_0\} \Rightarrow f(x) \neq y_0$

Allora $\lim_{x \to x_0} (g \circ f)(x) = \ell$.

Cioè $\lim_{x \to x_0} (g \circ f)(x) = \lim_{y \to y_0} g(y)$.

Es: calcoliamo $\lim_{x \to -\infty} \operatorname{arctg}(x^2)$.

è una composizione

$f(x) = x^2 \qquad g(y) = \operatorname{arctg} y$

$(g \circ f)(x) = g(f(x)) = g(x^2) = \operatorname{arctg}(x^2)$

$x_0 = -\infty \quad , \quad y_0 = \lim_{x \to x_0} f(x) = \lim_{x \to -\infty} x^2 = +\infty$

l'ipotesi 1) non è verificata perchè $y_0 = +\infty$
e non appartiene al dominio di g

ma l'ipotesi 2) è ovviamente verificata perché chiedo che $f(x) \neq y_0$ cioè $f(x) \neq +\infty$ che è ovviamente sempre vero.

Applico il teorema

$$\lim_{y \to y_0} g(y) = \lim_{y \to +\infty} \operatorname{arctg} y = \frac{\pi}{2}$$

$$\Rightarrow \lim_{x \to -\infty} \operatorname{arctg}(x^2) = \frac{\pi}{2}.$$

Oss: è un teorema di cambiamento di variabile.

$$\lim_{x \to -\infty} \operatorname{arctg}(x^2) \quad \Bigg| \quad \lim_{y \to +\infty} \operatorname{arctg} y = \frac{\pi}{2}.$$

cambio variabile e pongo $y = x^2$

se $x \to -\infty$ a quanto tende y?

$$\lim_{x \to -\infty} y = \lim_{x \to -\infty} x^2 = +\infty$$

Perchè si mette l'ipotesi 2) nel teorema?

Es: $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = 1 \quad \forall x \in \mathbb{R}$

$\boxed{x_0 = 0}$

$$g(x) = \begin{cases} 3 & \text{se } y = 1 \\ 5 & \text{se } y \neq 1 \end{cases}$$

$g : \mathbb{R} \to \mathbb{R}$.

$(g \circ f)(x) = g(f(x)) = g(1) = 3 \quad \forall x \in \mathbb{R}$

$\to \lim_{x \to 0} (g \circ f)(x) = 3$

ma $\lim_{y \to y_0} g(y) = \lim_{y \to 1} g(y) = 5$

$y_0 = \lim_{x \to x_0} f(x) = \lim_{x \to 0} f(x) = 1$

$\Rightarrow \lim_{x \to x_0} (g \circ f)(x) \neq \lim_{y \to y_0} g(y)$

ma non vale l'ipotesi ?)

e quando lo 1).

Limiti fondamentali

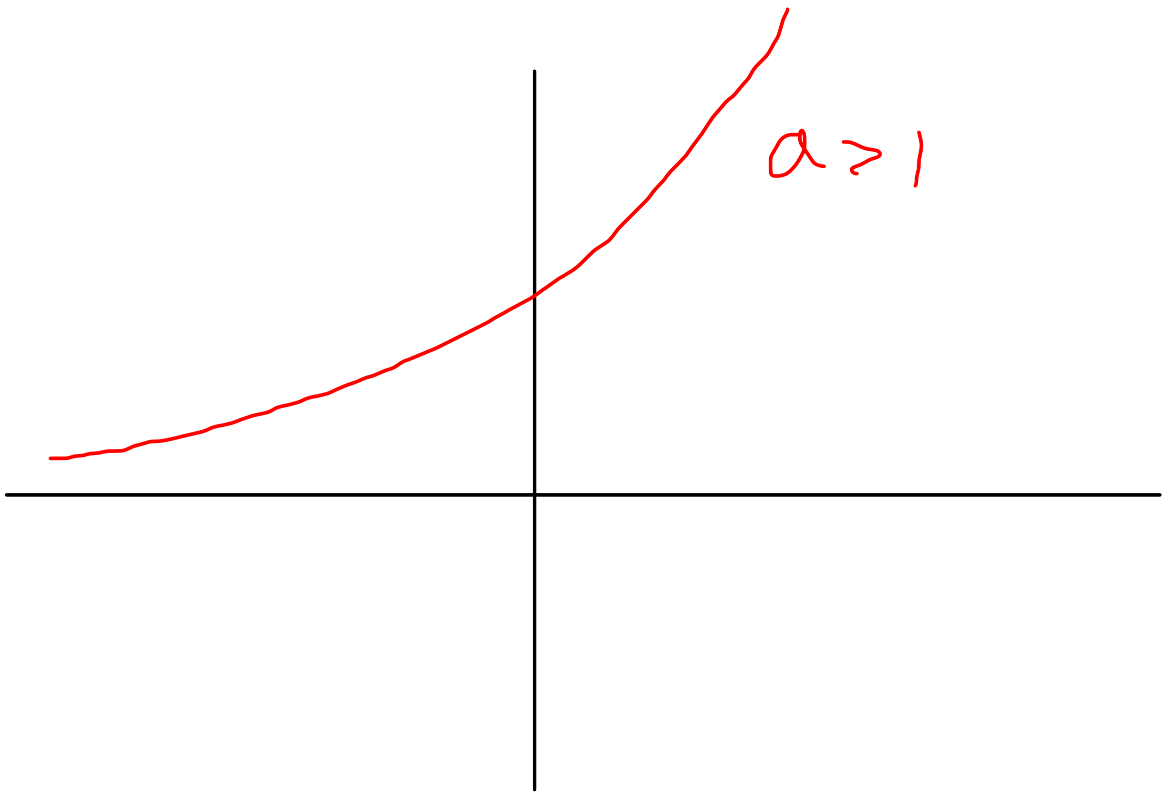
$$\lim_{x \to +\infty} a^x = \begin{cases} +\infty & \text{se } a > 1 \\ 1 & \text{se } a = 1 \\ 0^+ & \text{se } 0 < a < 1 \end{cases}$$

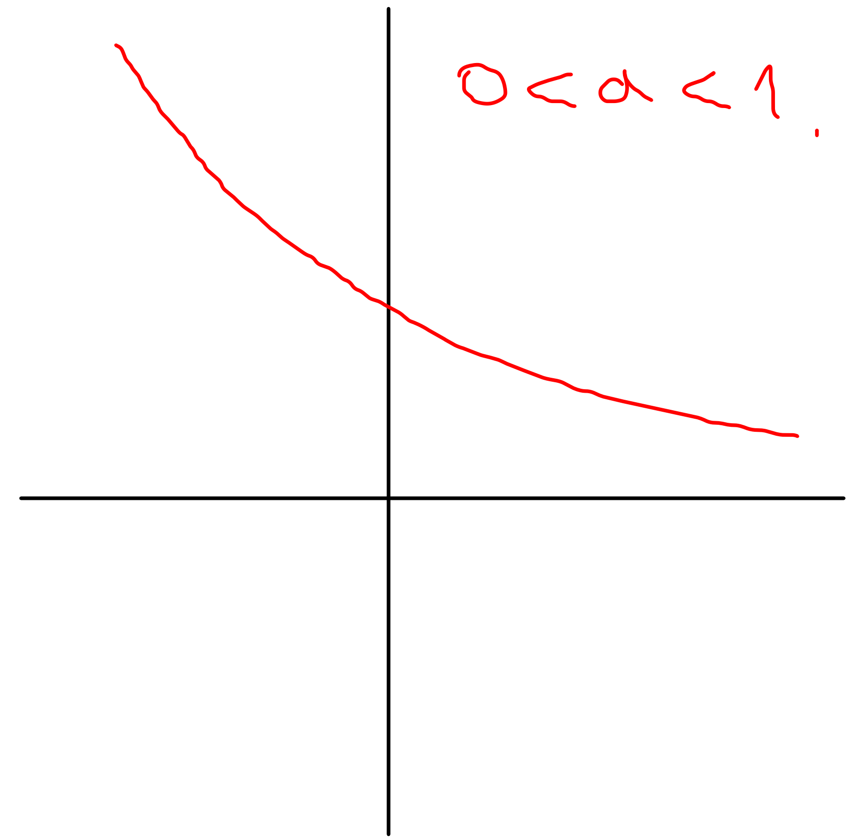
$$\lim_{x \to -\infty} a^x = \lim_{y \to +\infty} a^{-y} = \lim_{y \to +\infty} \frac{1}{a^y}$$

cambio variabile $y = -x$ quindi se $x \to -\infty$

allora $y \to +\infty$

$$\Rightarrow \lim_{x \to -\infty} a^x = \begin{cases} 0^+ & \text{se } a > 1 \\ 1 & \text{se } a = 1 \\ +\infty & \text{se } 0 < a < 1 \end{cases}$$

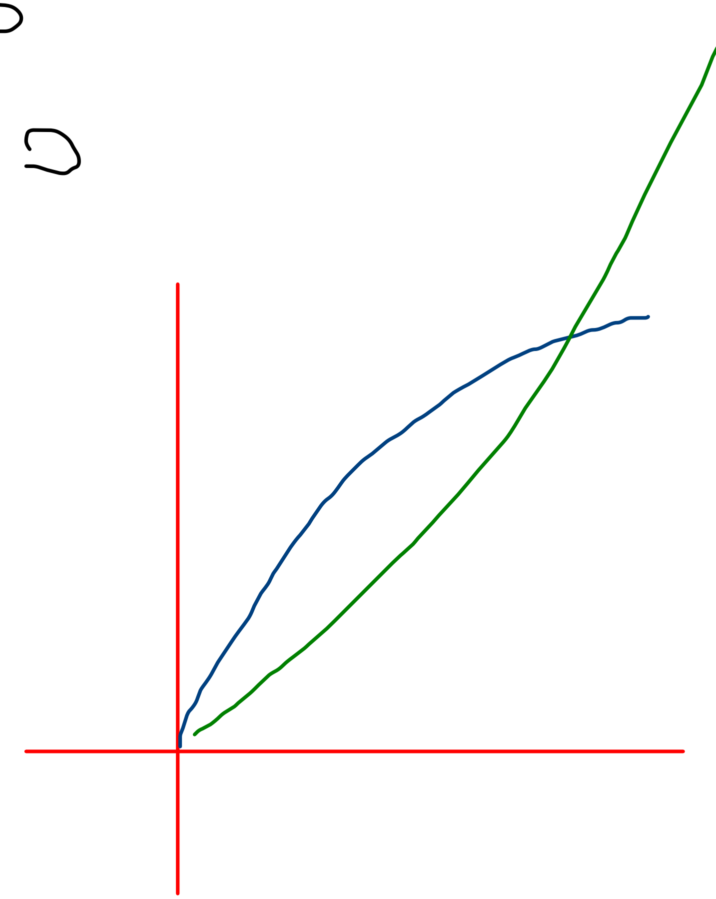




"Potenze" $\quad \alpha \in \mathbb{R} \quad x^\alpha$

$x > 0$

$$\lim_{x \to +\infty} x^\alpha = \begin{cases} +\infty & \text{se } \alpha > 0 \\ 1 & \text{se } \alpha = 0 \\ 0^+ & \text{se } \alpha < 0 \end{cases}$$



Confronti tra infiniti

$$\lim_{x\to+\infty} \frac{a^x}{x^\alpha} = \begin{cases} +\infty & \text{se } a>1 \\ 0^+ & \text{se } 0<a<1. \end{cases}$$

se $a=1 \Rightarrow a^x = 1$

$$\Rightarrow \lim_{x\to+\infty} \frac{a^x}{x^\alpha} = \lim_{x\to\infty} \frac{1}{x^\alpha}$$

caso precedente.

Es: $a = \frac{1}{2}$ $\qquad \alpha = -3$

$$\lim_{x \to +\infty} \frac{a^x}{x^\alpha} = \lim_{x \to +\infty} \frac{\left(\frac{1}{2}\right)^x}{x^{-3}} = \lim_{x \to +\infty} \boxed{\frac{x^3}{2^x}} = 0^+$$

$$\lim_{x \to +\infty} \frac{2^x}{x^3} = +\infty$$

Confronto fra logaritmo e potenze

$$\lim_{x \to +\infty} \frac{\log x}{x}$$

cambio di variabile

$y = \log x \implies x = e^y$

Se $x \to +\infty \implies y = \log x \longrightarrow +\infty$

$\left(\lim_{x \to +\infty} \log x = +\infty \right)$

$$\lim_{x \to +\infty} \frac{\log x}{x} = \lim_{y \to +\infty} \frac{y}{e^y} = 0$$

$$\lim_{x\to+\infty} \frac{(\log x)^\beta}{x^\alpha} = \circledast \qquad \alpha,\beta\in\mathbb{R},\ \alpha,\beta>0.$$

cambio di variabile $y = \lg x \Rightarrow x = e^y$

se $x\to+\infty \Rightarrow y\to+\infty$

$$\circledast = \lim_{y\to\infty} \frac{y^\beta}{(e^y)^\alpha} = \lim_{y\to\infty} \frac{y^\beta}{e^{y\cdot\alpha}} = \lim_{y\to\infty} \frac{y^\beta}{(e^\alpha)^y} =$$

$$= \lim_{y\to\infty} \frac{y^\beta}{a^y} = 0$$

$\boxed{\text{con } a = e^\alpha \text{ ma } \alpha>0 \Rightarrow a = e^\alpha > 1}$

$$\lim_{x \to 0^+} x \log x = 0 \cdot (-\infty) = \text{indeterminata.}$$

cambio di variabile $\quad y = \log x \quad , \quad x = e^y$

se $x \to 0^+ \Rightarrow y = \log x \longrightarrow -\infty$

$$\lim_{y \to -\infty} e^y \cdot y = e^{-\infty} \cdot (-\infty) = 0^+ (-\infty)$$

$$\uparrow$$

$$\lim_{y \to -\infty} e^y$$

indeterminata

$z = -y \quad$ se $y \to -\infty \Rightarrow z \longrightarrow +\infty$

$$\lim_{y \to -\infty} e^{y} \cdot y = \lim_{z \to +\infty} e^{-z}(-z) =$$

$$= \lim_{z \to +\infty} \frac{-z}{e^{z}} = 0$$

$$\boxed{\lim_{x \to 0^+} x \log x = 0}$$

$\boxed{\alpha > 0}$

$$\lim_{x \to 0^+} x^\alpha \log x$$

sostituisco
$y = x^\alpha$
quindi $x = y^{1/\alpha}$

se $x \to 0 \implies y = x^\alpha \to 0$

$$= \lim_{y \to 0^+} y \cdot \log\left(y^{1/\alpha}\right) =$$

$$= \lim_{y \to 0^+} y \cdot \frac{1}{\alpha} \log y = \frac{1}{\alpha} \cdot \lim_{y \to 0^+} y \log y = 0$$

Esercizio:

$$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = (1+0)^{\frac{1}{0^+}} = 1^{\infty} ?$$

$$(1+x)^{\frac{1}{x}} = e^{\log\left((1+x)^{\frac{1}{x}}\right)} = e^{\overbrace{\frac{1}{x}\log(1+x)}^{y}}$$

sostituzione $y = \frac{1}{x}\log(1+x)$

se $x \to 0^+$ $y \to ?$ $\lim_{x\to 0^+} \frac{1}{x}\log(1+x) = 1$

limite notevole visto prima.

$$\lim_{x \to 0^+} (1+x)^{\frac{1}{x}} = \lim_{y \to 1} e^y = e^1 = e.$$

$$\boxed{\lim_{x \to 0^+} (1+x)^{\frac{1}{x}} = e}$$

Nuovi casi di indeterminazione

$f(x) > 0$

$$\lim_{x \to x_0} f(x)^{g(x)}$$

quando è una forma indeterminata?

$$f(x)^{g(x)} = e^{\log\left(f(x)^{g(x)}\right)} = e^{\boxed{g(x)\log(f(x))}}$$

quando è indeterminato il limite

$$\lim_{x \to x_0} g(x) \cdot \log |f(x)| \quad ?$$

1) $\boxed{g \to 0}$, $\boxed{f \to +\infty}$ $\to \log(f) \to +\infty$

$0 \cdot \infty$ quindi $(+\infty)^0$ è indeterminata

2) $g \to 0$, $f \to 0^+$ $\Rightarrow$ $\log f \to -\infty$

$$\Rightarrow \quad g \cdot \log(f) = 0 \cdot (-\infty) = ?$$

$$\underbrace{(0^+)}_{f}{}^{\overset{g}{0}}$$

3) $g \to \pm\infty$, $f \to 1$ quidi $\log(f) \to 0$

$g \cdot \log(f) = \pm\infty \cdot 0 = ?$

$(1)^{\pm\infty}$ è indeterminata

$$(+\infty)^0,\ (0^+)^0,\ (1)^{+\infty},\ (1)^{-\infty}$$

sono forme indeterminate

Es:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\log(x^x)} = \lim_{x\to 0^+} e^{x\log x} = e^0 = 1$$

$y = x\log x$ se $x \to 0^+ \Rightarrow y \to 0$

$$\lim_{y\to 0} e^y = e^0 = 1.$$