Esercitazione 23/11

Esercizio 8

$$\int_0^{\frac{\pi}{2}} e^x \cos x \, dx$$

$$\int \underbrace{e^x}_{f} \cdot \underbrace{\cos x}_{g} \, dx = Fg - \int F g' \, dx = e^x \cos x - \int e^x(-\sin x) \, dx$$

$F = e^x$

$g' = -\sin x$

$$= e^x \cos x + \int \underbrace{e^x}_{f} \underbrace{\sin x}_{g} \, dx$$

$F = e^x$ $\quad g' = \cos x$

(provate a vedere cosa succede qui prendendo $f(x) = \sin x$ $g(x) = e^x$)

$$= e^x \cos x + \left(e^x \sin x - \int e^x \cos x \, dx \right)$$

$$= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$\Rightarrow 2 \int e^x \cos x \, dx = e^x(\cos x + \sin x)$$

Quindi $\displaystyle \int e^x \cos x \, dx = \frac{1}{2} e^x (\cos x + \sin x) + C$

$$\int_0^{\frac{\pi}{2}} e^x \cos x \, dx = \left[\frac{1}{2} e^x (\cos x + \sin x) \right]_0^{\frac{\pi}{2}} = \frac{1}{2} e^{\frac{\pi}{2}} \left(\cancel{\cos\left(\frac{\pi}{2}\right)} + \sin\left(\frac{\pi}{2}\right) \right) -$$

$$\frac{1}{2} e^0 \left(\cos(0) + \cancel{\sin(0)} \right)$$

$$= \frac{1}{2} e^{\frac{\pi}{2}} - \frac{1}{2} = \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right).$$

(a)

Esercizio 6

$F(x)$ la primitiva di $f(x) = \dfrac{\sin^2 x \cos x}{e + \sin^3 x}$

t.c. $F(0)=\frac{4}{3}$. Allora $F\left(\frac{\pi}{2}\right)=?$

Notiamo che $f(x)$ è definita su tutto $\mathbb{R}$, perché

$$-1 \leq \sin x \leq 1 \rightsquigarrow -1 \leq \sin^3 x \leq 1$$

$$\Rightarrow e+\sin^3 x \geq e-1>0.$$

Calcoliamo $\int f(x)\,dx$, poi imponiamo che $F(0)=\frac{4}{3}$,

e infine calcoliamo $F\left(\frac{\pi}{2}\right)$.

$$\int \frac{\sin^2 x \cos x}{e+\sin^3 x}\,dx \underset{\uparrow}{=} \int \frac{t^2}{e+t^3}\,dt \quad \text{e ora } (e+t^3)' = 3t^2$$

$t=\sin x$

$dt=\cos x\,dx$

$$= \frac{1}{3}\int \frac{3t^2}{e+t^3}\,dt \underset{\uparrow}{=} \frac{1}{3}\int \frac{dy}{y} =$$

$y=e+t^3$

$dy=3t^2\,dt$

$$= \frac{1}{3}\log|y| + C = \frac{1}{3}\log|e+t^3| + C$$

$$= \frac{1}{3}\log\underbrace{|e+\sin^3 x|}_{\substack{\text{il modulo si può togliere} \\ \text{perché } e+\sin^3 x>0 \\ \forall x\in\mathbb{R}.}} + C.$$

Se uno si accorgeva all'inizio che $(e+\sin^3 x)' = \underbrace{3\sin^2 x \cdot \cos x}_{\text{quasi il numeratore}}$

si poteva direttamente scrivere

$$\int \frac{\sin^2 x \cos x}{e+\sin^3 x}\,dx = \frac{1}{3}\int \frac{3\sin^2 x\cos x}{e+\sin^3 x}\,dx = \frac{1}{3}\log|e+\sin^3 x| + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C.$$

Ora cerchiamo la primitiva $F(x)$ t.c. $F(0)=\frac{4}{3}$

$$F(0) = \frac{1}{3}\log|e| + C = \frac{4}{3} \Rightarrow C = \frac{4}{3} - \frac{1}{3} = 1$$

$$\rightsquigarrow F(x) = \frac{1}{3}\log(e+\sin^3 x) + 1 .$$

Quindi $F\left(\frac{\pi}{2}\right) = \frac{1}{3}\log(e+1) + 1$. (c)

Esercizio 7 : $F(x) = \int_2^{x^3} e^{t^2}\,dt$.

Per rispondere calcoliamo la derivata seconda.

Sia $G(x) = \int_2^{x} e^{t^2}\,dt$. Allora $F(x) = G(x^3)$.

Il teo fond. del calcolo dice che $G'(x) = e^{x^2}$

Quindi derivando la funzione composta $F(x) = G(x^3)$.

ottengo

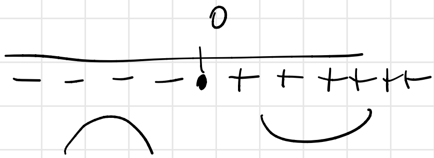
$$F'(x) = G'(x^3) \cdot (x^3)' = 3x^2 \cdot e^{x^6}$$

$$F''(x) = 6x \cdot e^{x^6} + 3x^2 \cdot 6x^5 \cdot e^{x^6}$$

$$= x e^{x^6} (6 + 18x^6)$$

Segno di $F''(x)$ è lo stesso di x, visto che e^{x^6} e $6+18x^6$ sono > 0 $\forall x \in \mathbb{R}$.

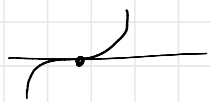
Quindi F''



Quindi $x = 0$ è l'unico punto di flesso di F.

(d)

($F'(x) \geq 0$ $\forall x$ e $= 0$ in $x = 0$



)

<u>Esercizio 9</u> : $F: \mathbb{R} \to \mathbb{R}$ $\quad F(x) = \displaystyle\int_1^x \frac{\arctan(t^2-1)}{1+t^2}\, dt$

Per il teo fond. del calcolo

$$F'(x) = \frac{\arctan(x^2-1)}{1+x^2}.$$

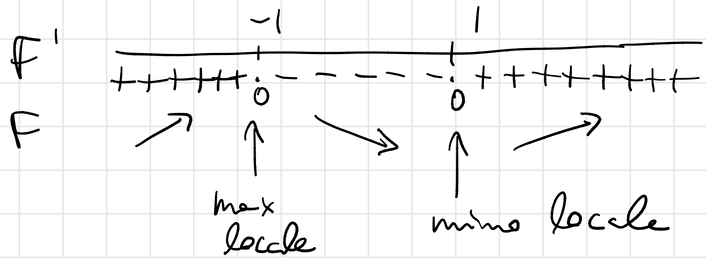
Studiamo il segno di $F'(x)$:

$F'(x) \geq 0 \iff \arctan(x^2-1) \geq 0$ $\quad (1+x^2 > 0 \ \forall x \in \mathbb{R})$

II

$x^2-1 \geq 0 \iff x \leq -1$ o $x \geq 1$

e vale $=0$ esattamente per $x = \pm 1$



(a)

Esercizio 5. $\displaystyle\int_{-2}^{-1} \frac{x+1}{x^2+2x+2}\,dx =$

Per integrali $\displaystyle\int \frac{ax+b}{cx^2+dx+e}\,dx$, il primo passaggio e' quello di modificare il numeratore per "trovarci" la derivata del denominatore.

In questo caso, calcoliamo $(x^2+2x+2)' = 2x+2$
$= 2\cdot\underbrace{(x+1)}_{\text{numeratore}}$

$$\frac{1}{2}\int_{-2}^{-1} \frac{2(x+1)}{x^2+2x+2}\,dx = \frac{1}{2}\Big[\log|x^2+2x+2|\Big]_{-2}^{-1}$$
$$= \frac{1}{2}\big(\log|1-2+2| - \log|4-4+2|\big)$$
$$= \frac{1}{2}\big(\log(1) - \log(2)\big) = -\frac{1}{2}\log(2)$$

$= \log \frac{1}{\sqrt{2}}$ (d)

Esempio più complicato:

$$\int \frac{3x+5}{x^2+2x+2}\,dx = \frac{3}{2}\int \frac{2x+\frac{5}{3}\cdot 2}{x^2+2x+2}\,dx =$$

$(x^2+2x+2)' = 2x+2$

$$= \frac{3}{2}\int \frac{2x+2+\left(\frac{5}{3}\cdot 2-2\right)}{x^2+2x+2}\,dx$$

$$= \frac{3}{2}\int \frac{\boxed{2x+2}}{x^2+2x+2}\,dx + \frac{3}{2}\int \frac{\frac{10-6}{3}}{x^2+2x+2}\,dx$$

$$= \frac{3}{2}\log|x^2+2x+2| + \frac{3}{2}\cdot\frac{4}{3}\cdot\int \frac{1}{x^2+2x+2}\,dx$$

$x^2+2x+2=0$ non ha radici

$x^2+2x+2 = (x^2+2x+1)+1$

$= (x+1)^2+1$

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx = \arctan(x+1) + C$$

(sostituzione $y = x+1$, $dy = dx$ vi da $\int \frac{1}{y^2+1}\,dy = \arctan(y) + c$)

Esercizio 1: $\int\left(\frac{2x}{x^2+2}+1\right)dx = \int \frac{2x}{x^2+2}\,dx + \int 1\,dx$

$(x^2+2)' = 2x$

$$= \log\underbrace{|x^2+2|}_{\text{si può togliere } |\cdot| \text{ perché } x^2+2>0 \ \forall x \in \mathbb{R}} + x + C$$

$$= \log(x^2+2) + x + C. \qquad (d)$$

(fare la verifica derivando!)

Esercizio 2 : $\displaystyle\int_0^{\frac{\sqrt{\pi}}{2}} x\sin(x^2)\cos(x^2)\,dx =$

$$\int x\sin(x^2)\cos(x^2)\,dx = \int x\cdot\frac{1}{2}\sin(2x^2)\,dx = \frac{1}{2}\int x\cdot\sin(2x^2)\,dx =$$

usiamo

$\sin(2t) = 2\sin(t)\cos(t)$

con $t = x^2$

$(2x^2)' = 4x$

$$= \frac{1}{8}\int 4x\cdot\sin(2x^2)\,dx$$

$$= \frac{1}{8}\left[-\cos(2x^2)\right] + C$$

(Check: $(-\cos(2x^2))' = \sin(2x^2)\cdot 4x$)

Quindi $\displaystyle\int_0^{\frac{\sqrt{\pi}}{2}} x\cdot\sin(x^2)\cos(x^2) = \frac{1}{8}\left[-\cos(2x^2)\right]_0^{\frac{\sqrt{\pi}}{2}}$

$$= \frac{1}{8}\left(-\cos\left(2\cdot\frac{\pi}{4}\right) + \cos(0)\right)$$

$$= \frac{1}{8}\left(\cos(0)\right) = \frac{1}{8} \qquad (a).$$

Alternativamente

$$\int \underbrace{x\sin(x^2)}_{f} \cdot \underbrace{\cos(x^2)}_{g}\, dx = -\frac{1}{2}\cos(x^2)\cdot\cos(x^2) - \int + \frac{1}{\cancel{2}}\cos(x^2)\cdot \cancel{2}x\cos(x^2)\, dx$$

$F = \int x\sin(x^2)\,dx$ $\qquad g' = -2x\sin(x^2)$

$= \frac{1}{2}\int 2x\sin(x^2)\,dx$

$= \frac{1}{2}\left(-\cos(x^2)\right)$

$$= -\frac{1}{2}\cos^2(x^2) - \int x\cos(x^2)\sin(x^2)\,dx$$

$$\Longrightarrow \int x\sin(x^2)\cos(x^2)\,dx = -\frac{1}{4}\cos^2(x^2)$$

(differisce da quello che ho scritto io per una costante)

$$-\frac{1}{8}\cos(2x^2) = -\frac{1}{8}\left(2\cos^2(x^2) - 1\right)$$

$$\shortparallel$$

$$-\frac{1}{4}\cos^2(x^2) + \frac{1}{8}$$

$\cos(2t) = \cos^2 t - \sin^2 t$

$= 2\cos^2 t - 1$

Esercizio 3.

$$\int_0^1 e^x(2x+1)\,dx$$

$$\int e^x(2x+1)\,dx = 2\int e^x x\,dx + \int e^x\,dx =$$

$$= 2\int \underbrace{x}_{g}\underbrace{e^x}_{f}\,dx + e^x$$

$g' = 1 \qquad F = e^x$

$$= 2\left(e^x \cdot x - \int e^x \cdot 1\,dx\right) + e^x$$

$$= 2xe^x - 2e^x + e^x = 2xe^x - e^x + C$$

(verifico!)

$$\int_0^1 e^x(2x+1)\,dx = \left[2xe^x - e^x\right]_0^1 = 2e - e - 2\cdot 0\cdot e^0 + e^0$$

$$= e + 1 \quad (c)$$

Esercizio 4 : $\lim_{x\to+\infty} \int_0^x \frac{t^2}{1+t^2}\,dt$

$$\int \frac{t^2}{1+t^2}\,dt = \int \frac{1+t^2-1}{1+t^2}\,dt = \int 1\,dt - \int \frac{1}{1+t^2}\,dt$$

$$= t - \arctan(t) + C$$

(divisione

$$\begin{array}{rr|l} t^2+0\cdot t+0 & & 1+t^2 \\ \hline -t^2 -1 & & 1 \text{ quoziente} \\ \hline // \quad \underbrace{0\cdot t-1}_{\text{resto}} & & \end{array}$$

$t^2 = 1\cdot(t^2+1) - 1$ (che è quello che ho scritto sopra))

Quindi $\int_0^x \frac{t^2}{1+t^2}\,dt = \left[t - \arctan(t)\right]_0^x =$

$$= x - \arctan(x) - 0 + \arctan(0)$$

$$= x - \arctan(x)$$

$$\lim_{x\to+\infty} = \lim_{x\to+\infty}\left(x - \arctan(x)\right) = +\infty - \frac{\pi}{2} = +\infty.$$

(d)

Interpretazione geometrica:

(domanda: si poteva sostituire $y=t^2$?

$$\int \frac{t^2}{1+t^2}\,dt = \int \frac{y}{1+y}\cdot\frac{dy}{2\sqrt{y}} = \int \frac{\sqrt{y}}{1+y}\,dy \quad ??$$

$t=\sqrt{y}$ $\rightsquigarrow$ $y=t^2$

$(y\geq 0)$

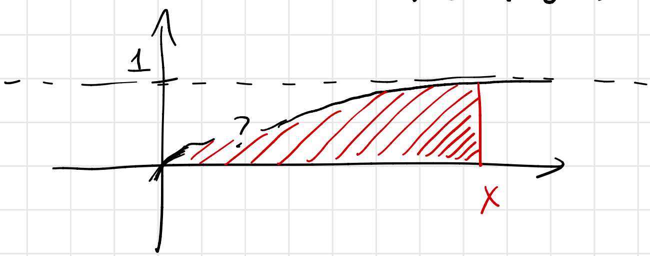
$$dy = 2t\,dt \rightsquigarrow dt = \frac{dy}{2t} = \frac{dy}{2\sqrt{y}}$$)

---

la funzione $f(t)=\frac{t^2}{1+t^2} \xrightarrow{t\to+\infty} 1$ dal basso, perché $1+t^2 > t^2$.

$f(0)=0$.

$\int_0^x$ = area ( )



Ad esempio, visto che $\frac{t^2}{1+t^2}\to 1$ per $t\to+\infty$,

da un certo t_0 in poi si avrà $\frac{t^2}{1+t^2}\geq\frac{1}{2}$.

Quindi

$$\int_0^x \frac{t^2}{1+t^2}\,dt = \underbrace{\int_0^{t_0}}_{\substack{\text{numero} \\ = A}} + \int_{t_0}^x \qquad (\text{per } x\geq t_0)$$

$$= A + \int_{t_0}^x \frac{t^2}{1+t^2}\,dt$$

$$\geq A + \int_{t_0}^x \frac{1}{2}\,dt$$

$$= A + \frac{1}{2}(x - t_0)$$

e quindi $\lim_{x\to+\infty}\left(\int_0^x \frac{t^2}{1+t^2}\,dt\right) \geq \lim_{x\to+\infty}\left(A + \frac{1}{2}(x-t_0)\right)$

$$= +\infty$$

Esercizio 10 : $\lim_{a\to+\infty}\int_{-a}^{a} x e^{-x}\,dx$

$$\int \underbrace{x}_{g}\,\underbrace{e^{-x}}_{f}\,dx = -e^{-x}\cdot x - \int -e^{-x}\cdot 1\,dx$$

$g' = 1 \quad F = -e^{-x}$

$$= -xe^{-x} + \int e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x} = -e^{-x}(x+1) + C$$

Verifica: $\left(-e^{-x}(x+1)\right)' = e^{-x}(x+1) - e^{-x}(1)$

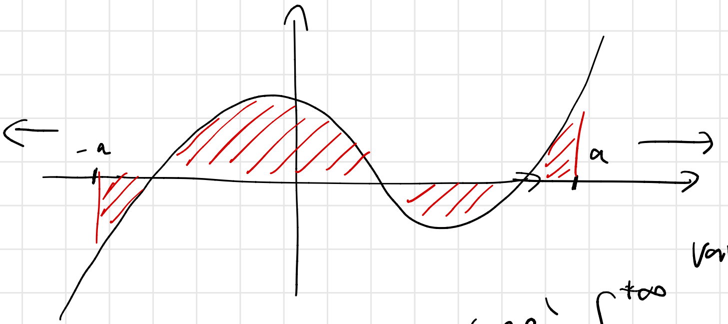
$$= x\cdot e^{-x} + \cancel{e^{-x}} - \cancel{e^{-x}} = xe^{-x}.$$

$$\int_{-a}^{a} x e^{-x}\,dx = \left[-e^{-x}(x+1)\right]_{-a}^{a} = -e^{-a}(a+1) + e^{a}(-a+1)$$

$$\lim_{a\to+\infty} = \lim_{a\to+\infty}\left(\underbrace{-\underbrace{e^{-a}}_{\downarrow\,0}\underbrace{(a+1)}_{\downarrow\,+\infty}}_{\to 0} + \underbrace{\underbrace{e^{a}}_{\downarrow\,+\infty}\cdot\underbrace{(-a+1)}_{\downarrow\,-\infty}}_{\to -\infty}\right) = -\infty \qquad \text{(d)}.$$

$$\lim_{a\to+\infty} e^{-a}(a+1) = 0$$

perché $\lim_{a\to+\infty}\frac{a+1}{e^a} \overset{\text{H\^op}}{=} \lim_{a\to+\infty}\frac{1}{e^a} = 0.$



$\underline{\underline{\text{non}}}$ è $\int_{-\infty}^{\infty}$!!

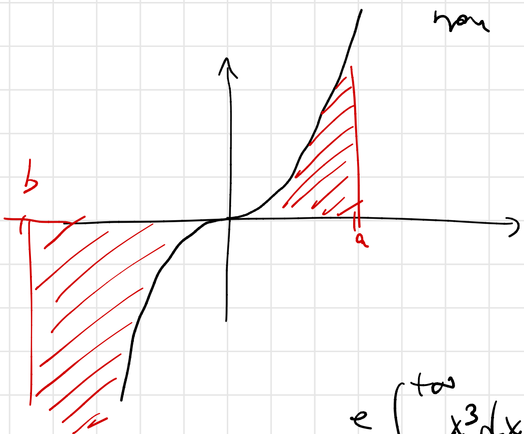
Perché i due limiti vanno fatti "separatamente"

cioè $\int_{-\infty}^{+\infty} = \int_{-\infty}^{0} + \int_{0}^{+\infty}$

se entrambi ↑ hanno senso e la somma non è indeterminata.

<u>Esempio</u>:

$f(x)=x^3$



$\int_0^{+\infty} x^3 dx = +\infty$

e $\int_{-\infty}^{0} x^3\, dx = -\infty$

e $\int_{-\infty}^{+\infty} x^3 dx = \int_{-\infty}^{0} x^3 dx + \int_0^{+\infty} x^3 dx$

$= -\infty + \infty$??

non esiste

(x^3 non è integrabile in senso generalizzato su $(-\infty, +\infty)$.)