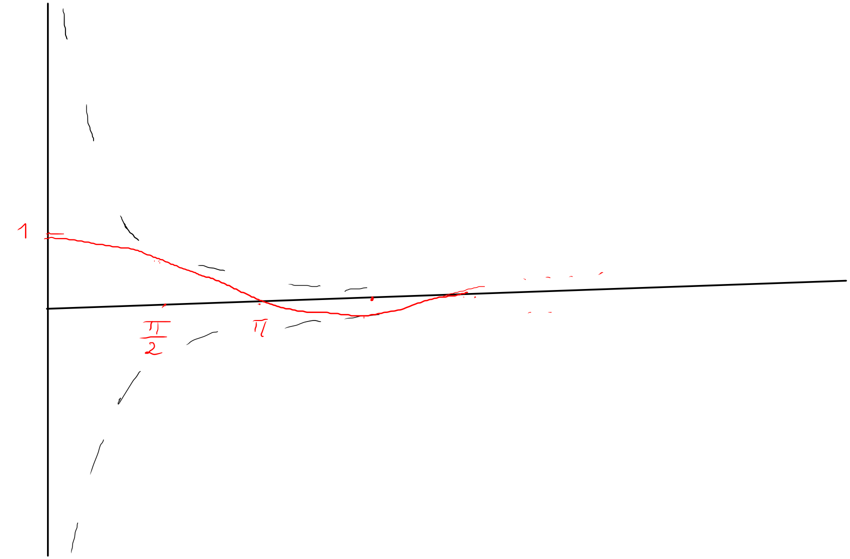
$$\exists \int_1^{+\infty} \frac{\sin x}{x}\,dx \in \mathbb{R}$$



$$\sin(x+2\pi) = \sin x$$

$$\int_c^{c+2\pi} (\sin x)\,dx = 0$$

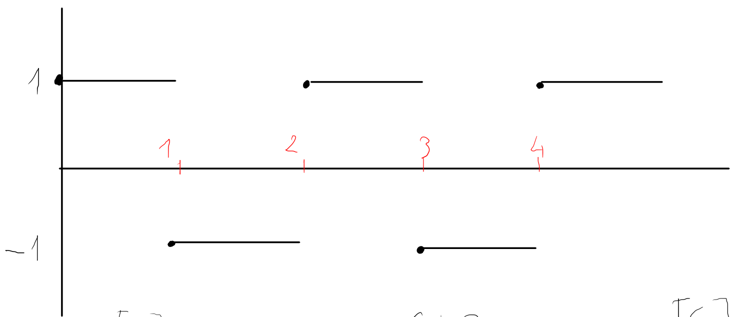
$$\frac{1}{x} \xrightarrow{x\to+\infty} 0$$

Si esamina un caso analogo piuttosto concreto in cui si capisce la validità generale di

$f \searrow 0$, $g(x+T)=g(x)$ $\displaystyle\int_0^T g(x)\,dx = 0 \Longrightarrow \exists \int^{+\infty} f(x)g(x)\,dx \in \mathbb{R}$

f e g Riem. int sugli intervalli $[a;b]$

$$s(x) = \begin{cases} 1 & 2n \le x < 2n+1 \\ -1 & 2n+1 \le x < 2n+2 \end{cases} = (-1)^{[x]} \qquad x \geq 0$$



$s(x) = s(x+2)$

è 2-periodica

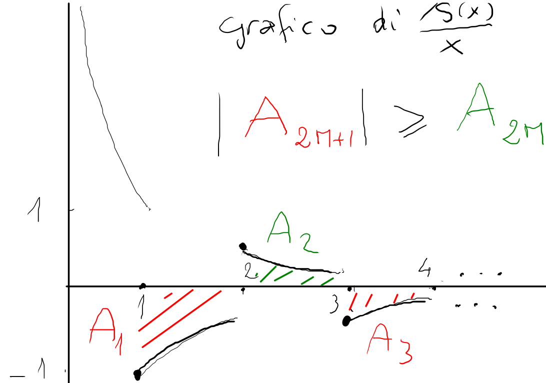
$$\int_c^{c+2} s(x)\,dx =$$

$$= \int_c^{[c]+2} s(x)\,dx + \int_{[c]+2}^{c+2} s(x)\,dx \overset{y=x-2}{=} \int_c^{[c]+2} s(x) + \int_{[c]}^{c} s(y)\,dy = \int_{[c]}^{[c]+2} s(x)\,dx = 0$$

? $\exists \lim_{N\to+\infty} \int_1^N \frac{s(x)}{x}\,dx$, nel caso è finito ?

grafico di $\frac{s(x)}{x}$

$$|A_{2M+1}| \geq A_{2M+2}$$



Ci si aspetta che se esiste

$$\int_1^{+\infty} \frac{s(x)}{x}\,dx < 0$$

in quanto le aree "negative" son più estese di quelle "positive" subito successive

① Si esamina il caso in cui $N = 2M+1$ con $M \in \mathbb{N}$.

$$\int_1^{2M+1} \frac{s(x)}{x}\,dx = \int_1^2 -\frac{1}{x}\,dx + \int_2^3 \frac{1}{x}\,dx + \dots + \int_{2h-1}^{2h} -\frac{1}{x}\,dx + \int_{2h}^{2h+1} \frac{1}{x}\,dx + \dots + \int_{2M}^{2M+1} \frac{1}{x}\,dx$$

$$= -(\log 2 - \log 1) + (\log 3 - \log 2) + \dots - (\log 2h - \log(2h-1)) + (\log 2h+1 - \log 2h) + \dots$$

$$= -\log 2 + \log \frac{3}{2} + \dots - \log \frac{2h}{2h-1} + \log \frac{2h+1}{2h} + \dots - \log \frac{2M}{2M-1} + \log \frac{2M+1}{2M}$$

$$= \log \frac{3}{4} + \log \frac{15}{16} + \dots + \log \frac{4h^2-1}{4h^2} + \dots + \log \frac{4M^2-1}{4M^2}$$

$$\int_1^{2M+1} \frac{S(x)}{x}\,dx = \sum_{h=1}^{M} \log\left(\frac{4h^2-1}{4h^2}\right) = \sum_{h=1}^{M} \log\left(1-\frac{1}{4h^2}\right)$$

resto di Lagrange

$\log(1+t) =$
$= t - \frac{1}{2(1+\xi)^2}t^2$

ξ tra t e 0

$$= -\sum_{h=1}^{M} \frac{1}{4h^2} + \frac{1}{2(1+\xi_h)^2}\cdot\frac{1}{16h^2} \underset{\text{def.}}{=} -S_M < 0$$

S_M è crescente quindi ha sempre limite

poiché $-\frac{1}{4} \overset{h\geq 1}{\leq} -\frac{1}{4h^2} < \xi_h < 0$ si ha $1 < \frac{1}{(1+\xi)^2} < \frac{16}{9}$

$$\exists \lim_{M\to\infty} -\int_1^{2M+1} \frac{S(x)}{x}\,dx = \lim_{M\to\infty} S_M \leq \left(\frac{1}{4}+\frac{1}{2}\frac{16}{9}\frac{1}{16}\right)\cdot\lim_{M\to+\infty}\sum_{h=1}^{M}\frac{1}{h^2}$$

ma $$\sum_{h=1}^{M}\frac{1}{h^2} = \int_1^{M}\frac{1}{[x]^2}\,dx \overset{\text{C.A.}}{\sim} \int_1^{M}\frac{1}{x^2}\,dx \xrightarrow[M\to\infty]{} 1$$

$$\frac{x^2}{[x]^2} \xrightarrow[x\to+\infty]{} 1$$

Quindi $\exists \lim\limits_{\substack{M \to +\infty \\ M \in \mathbb{N}}} \int_1^{2M+1} \frac{s(x)}{x}\, dx \underset{\neq}{>} -\infty$

2

$$\int_1^{2M} \frac{s(x)}{x}\, dx = \int_1^{2M+1} \frac{s(x)}{x}\, dx - \int_{2M}^{2M+1} \frac{1}{x}\, dx =$$

$$= \int_1^{2M+1} \frac{s(x)}{x}\, dx - \log\left(1 + \frac{1}{2M}\right) \xrightarrow{M \to +\infty} 0$$

3 Caso generale $N \geq 1,\ N \in \mathbb{R}$

$$\int_1^{N} \frac{s(x)}{x}\, dx = \int_1^{[N]} \frac{s(x)}{x}\, dx \pm \int_{[N]}^{N} \frac{1}{x}\, dx = \int_1^{[N]} \frac{s(x)}{x}\, dx \pm \log\frac{N}{[N]}$$

$$\log\frac{N}{[N]} \xrightarrow{N \to +\infty} 0$$

[redacted]