Exam practice test

Numerical Methods and Optimization course
University of Pisa, 2016-12-13

You may use Matlab, pencil or paper, or a calculator (unless explicitly stated in the exercise). You may use the quick reference sheet on Matlab's syntax posted on the web page of the course.

Exercise 1. Consider the following constrained optimization problem:

$$\begin{cases} \min\ x_1^2 + x_2^2 + x_3^2 - 2x_1 - 2x_2 - 2x_3 \\ x_1^2 + x_2^2 + x_3^2 - 1 \leq 0 \\ -x_3 \leq 0 \end{cases}$$

a. Do global optimal solutions exist? Why?

b. Is it a convex problem? Why?

c. Do constraint qualifications hold in any feasible point?

d. Is the point $(1, 0, 0)$ a local minimum? Why?

e. Find all the solutions of the KKT system.

f. Find local minima and global minima.

g. Find the objective function and constraints of the Lagrangian dual problem.

h. Is $\lambda = (0, 1)$ an optimal solution of the Lagrangian dual problem? Why?

Exercise 2. Consider the following unconstrained optimization problem:

$$\begin{cases} \min\ f(x) = 2x_1^2 + 3x_2^2 + x_3^2 + 2x_1x_2 + x_2x_3 - x_1 - 3x_2 - 2x_3 - \log(x_1 + x_2 + x_3) \\ x \in \text{dom}(f) \end{cases}$$

a. Is it a convex problem? Why?

b. Do global minima exist? Why?

c. Is the global minimum unique? Why?

d. Solve the problem by means of the gradient method with inexact line search setting $\alpha = 0.1$, $\gamma = 0.9$, $\bar{t} = 1$, starting from the point $(10, 10, 10)$ and using $\|\nabla f(x)\| < 10^{-6}$ as stopping criterion. Which is the global minimum? How many iterations are needed? Which is the optimal value?

e. Solve the problem by means of the Newton method with inexact line search setting $\alpha = 0.1$, $\gamma = 0.9$, $\bar{t} = 1$, starting from the point $(10, 10, 10)$ and using $\|\nabla f(x)\| < 10^{-6}$ as stopping criterion. Which is the global minimum solution? How many iterations are needed? Which is the optimal value?

Exercise 3. a. Generate pseudorandom matrices with the Matlab commands

```
rng(0); A = randn(20,8);
b = randn(20, 1); v = randn(20, 1);
```

Set $M = \begin{bmatrix} A & v \end{bmatrix}$. Compute the solution x of the least-squares problem $\min\|Mx - b\|_2$ using a thin QR factorization $M = Q_1 R_1$ (you may use Matlab's function `qr` to obtain it). What is the computed value of the residual $\|Mx - b\|_2$?

b. What is the value of the condition number $\kappa(M)$ in this example? Based on this condition number (and your knowledge), do you judge the computed result to be a good approximation of the solution?

c. Show that the thin QR factorization $A = \hat{Q}_1 \hat{R}_1$ of A can be obtained from the values Q_1, R_1 that you have already computed, without need for further computations.

Write a Matlab `function [x, y] = doublels(A, v, b)` that returns the solutions of the two least squares problems $\min\|Mx - b\|_2$ and $\min\|Ay - b\|_2$. Try to avoid superfluous operations: for instance, obtain the thin QR factorization of A from that of M, as indicated above.

Report on paper the code of the function.

d. Can you show that for each choice of A, v, b the inequality $\min\|Mx - b\|_2 \leq \min\|Ay - b\|_2$ always holds, i.e., the solution of the LS problem with matrix M always has a smaller residual than the one with matrix A?
Hint: can you find a vector z such that $Mz = Ay$?

e. Now we want to do the reverse: obtain a QR factorization of M from one of A. For simplicity, we will consider full QR factorizations instead of the thin version.

Show that, given the factors $\hat{Q} \in \mathbb{C}^{m \times m}, \hat{R} \in \mathbb{C}^{m \times n}$ of the QR factorization of A, one can construct in time $O(m^2)$ matrices $H \in \mathbb{C}^{m \times m}, R \in \mathbb{C}^{m \times (n+1)}$ such that the QR factorization of M is $(\hat{Q}H)R$.
Hint: which entries of the matrix $Q^* M$ are zero? How can we transform $Q^* M$ into a triangular matrix?