Lesson 7
Markov Chains
Modelling Chord Routing

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Outline

1. Basic Definitions
2. Examples
3. It’s All Just Matrix Theory?
4. The Basic Theorem
5. Modelling Chord Routing by Markov Chains
Markov Chain: Basic Characteristics

- Markov Chain: describes a system whose states change over time
  - discrete time stochastic process
- Changes are governed by a probability distribution.
- The next state only depends upon the current system state
  - the path to the present state is not relevant
- Class of random process useful in different areas
  - developments in theory and applications in recent decades.
Markov Chain Specification

A sequence of Random Variables \( \{X_0, X_1, \ldots X_n\} \): \( X_i \) describes the state of the system at time \( i \)

To specify a Markov Chain we need:

- a finite or countable set of states \( S \)
  - \( S = \{1, 2, \ldots, N\} \) for some finite \( N \)
  - the value of the random variables \( X_i \) are taken from \( S \)

- an initial distribution \( \pi_0 \), \( \pi_0(i) = \mathbb{P}\{X_0 = i\} \): probability that the Markov Chain starts out in state \( i \)

- the probability transition rules
The Probability Matrix

- The probability matrix $P = (P_{ij})$ specifies the transition rules.
- If the size of $S$ is $N$, $P$ is a $N \times N$ stochastic matrix:
  - Each entry is non-negative.
  - The sum of each row is 1.
- $P_{ij}$ is a conditional probability: defines the probability that the chain jumps to state $j$, at time $n+1$, given that it is in state $i$ at time $n$,
  $$P_{ij} = P(X_{n+1} = j \mid X_n = i)$$
- We assume *Time Homogeneity*: the probability does not depend on the time $n$, but probability only on depends state $i$ and $j$. 
Time Homogeneity

- "where I go next given that I am in state $s$ at time $x$ is equal to where I go next given that I am in the same state $s$ at time $y \neq x"$

- every time the chain is in state $s$, the probability of jumping to another state is the same

- the same probability over time

- we will assume time homogeneity in the following
The Markov Frog

A frog hopping among lily pads

- State Space: $S = \{1, 2, 3\}$ represent the pads
- Initial Distribution: $\pi_0 = (1/2, 1/4, 1/4)$
- Probability Transition Matrix:

$$P = \begin{pmatrix}
0 & 1 & 0 \\
1/3 & 0 & 2/3 \\
1/3 & 1/3 & 1/3
\end{pmatrix}$$

- defines the probabilities of jumping from one state to another one
The Markov Frog

- the frog chooses its initial position $X_0$ according to the initial distribution $\pi_0$

- to this purpose, the frog can ask its computer to generate a uniformly distributed random number $U_0$ in the interval $I = [0, 1]$ and then taking

$$X_0 = \begin{cases} 
1 & \text{if } 0 \leq U_0 \leq 1/2 \\
2 & \text{if } 1/2 < U_0 \leq 3/4 \\
3 & \text{if } 3/4 < U_0 \leq 1 
\end{cases}$$

- for instance, if $U_0 = 0.8419$ then $X_0 = 3$
  - the frog starts on the third lily pad
The Markov Frog

- Evolution of the Markov Chain: the frog chooses a lily pad to jump
  - state after the first jump = value of the random variable $X_1$
- the frog starts from lily pad 3 so look at the probability distribution in row 3 of $P$, namely $(1/3,1/3,1/3)$
- again, generate a uniformly distributed random number $U_1$ in the interval $I = [0, 1]$ then takes

\[ X_1 = \begin{cases} 
1 & \text{if } 0 \leq U_1 \leq 1/3 \\
2 & \text{if } 1/3 < U_1 \leq 2/3 \\
3 & \text{if } 2/3 < U_1 \leq 1 
\end{cases} \]

- if $U_1 = 0.1234$, then $X_1 = 1$
  - the frog jumps from the lily pad 3 to lily pad 1
  - $X_1 = 1$ there is no choice for the value of $X_2$, it must be 2
  - and so on....
The Markov Property

- in the previous example
  \[ P\{X_3 = j \mid X_2 = 2, X_1 = 1, X_0 = 3\} = P\{X_3 = j \mid X_2 = 2\} \quad \forall j \]
- the only information relevant to the distribution to \( X_3 \) is the information that \( X_2 = 2 \)
- \( X_0 = 3 \) and \( X_1 = 1 \) may be ignored!

**Definition**

A stochastic process \( X_0, X_1, \ldots \) satisfies the Markov Property if

\[
P\{X_{n+1} = i_{n+1} \mid X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} = P\{X_{n+1} = i_{n+1} \mid X_n = i_n\}
\]
Markov Chains: Ehrenfest Chain

There is a total of 6 balls in two urns, 2 in the first and 4 in the second. We pick one of the 6 balls at random and move it to the other urn.

\( X_n \) number of balls in the first urn, after the \( n \)th move.
Markov Chains: Ehrenfest Chain

\[ P(X_0 = 2) = 1 \]

\[ P(X_1 = j) = \begin{cases} 
\frac{4}{6} & j = 3 \\
\frac{2}{6} & j = 1 \\
0 & \text{otherwise}
\end{cases} \]

\[ P(X_{n+1} = l \mid X_n = j) = \begin{cases} 
\frac{j}{6} & l = j - 1 \\
\frac{(6 - j)}{6} & l = j + 1 \\
0 & \text{otherwise}
\end{cases} \]

In 5 units of time \( X_0, \ldots X_5 \) might follow the following path:

\[ S = 4, \quad X_1 = 3, \quad X_2 = 2, \quad X_3 = 4, \quad X_4 = 3, \quad X_5 = 2 \]
Ehrenfast Chain: Probability Transition Matrix

\[ P = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1/6 & 0 & 5/6 & 0 & 0 & 0 & 0 \\
0 & 2/6 & 0 & 4/6 & 0 & 0 & 0 \\
0 & 0 & 3/6 & 0 & 3/6 & 0 & 0 \\
0 & 0 & 0 & 4/6 & 0 & 2/6 & 0 \\
0 & 0 & 0 & 0 & 5/6 & 0 & 1/6 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/0
\end{pmatrix} \]
given a graph and a starting point

select one of its neighbours at random, and move to this neighbor

then select a neighbour of the new point at random, and move to it, and so on . . .

the (random) sequence of points selected in this way is a *random walk* on the graph.

there is a strict relation between Random Walk and Markov chains

* time-reversible Markov chains can be viewed as random walks on undirected graphs
* we will see this later . . .
what is the probability $p_{ij}^n$ that, given the chain in state $i$, it will be in state $j$, $n$ step after?

if we start on lily pad 3, what is the probability of being on lily pad 1, after 2 steps?

$$p_{31}^2 = p_{31}p_{11} + p_{32}p_{21} + p_{33}p_{31} = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$$

This is the dot product third row-first column of $P$.

this returns the 1,3-entry of the product of $P$ with itself.
The generalization of the previous observation leads to the following theorem.

**Theorem**

Let $P$ be the transition matrix of a Markov chain. The $ij$-th entry of the matrix $P^n$ gives the probability that the Markov chain, starting in state $s_i$, will be in state $s_j$ after $n$ steps.

The power of the transition matrix gives interesting information about the evolution of the process.
It’s All Just Matrix Theory?

\[ P^1 = \begin{pmatrix}
  pad_1 & pad_2 & pad_3 \\
  pad_1 & 0 & 1 & 0 \\
  pad_2 & 0.333 & 0 & 0.666 \\
  pad_3 & 0.333 & 0.333 & 0.333 \\
\end{pmatrix} \]

\[ P^2 = \begin{pmatrix}
  pad_1 & pad_2 & pad_3 \\
  pad_1 & 0.333 & 0 & 0.666 \\
  pad_2 & 0.221 & 0.554 & 0.221 \\
  pad_3 & 0.221 & 0.443 & 0.332 \\
\end{pmatrix} \]

\[ P^3 = \begin{pmatrix}
  pad_1 & pad_2 & pad_3 \\
  pad_1 & 0.258 & 0.295 & 0.442 \\
  pad_2 & 0.244 & 0.404 & 0.342 \\
  pad_3 & 0.244 & 0.392 & 0.355 \\
\end{pmatrix} \]

\[ P^4 = \begin{pmatrix}
  pad_1 & pad_2 & pad_3 \\
  pad_1 & 0.246 & 0.368 & 0.371 \\
  pad_2 & 0.244 & 0.369 & 0.367 \\
  pad_3 & 0.245 & 0.369 & 0.367 \\
\end{pmatrix} \]
### It’s All Just Matrix Theory?

**Example 1:**

$$P^5 = \begin{pmatrix}
  \text{pad}_1 & \text{pad}_2 & \text{pad}_3 \\
  0.241 & 0.363 & 0.362 \\
  0.239 & 0.361 & 0.360 \\
  0.240 & 0.361 & 0.361
\end{pmatrix}$$

**Example 2:**

$$P^6 = \begin{pmatrix}
  \text{pad}_1 & \text{pad}_2 & \text{pad}_3 \\
  0.231 & 0.249 & 0.248 \\
  0.230 & 0.247 & 0.246 \\
  0.230 & 0.247 & 0.247
\end{pmatrix}$$

**Example 3:**

$$P^7 = \begin{pmatrix}
  \text{pad}_1 & \text{pad}_2 & \text{pad}_3 \\
  0.213 & 0.322 & 0.321 \\
  0.212 & 0.320 & 0.320 \\
  0.212 & 0.321 & 0.320
\end{pmatrix}$$
The probabilities of the three lily pads are .2, .3, and .3 no matter where the frog starts at the first step.

the long range predictions are independent from the starting state

the columns are about identical because the chain forgets the initial state

but ... this does not happen for all chains ... 

we will show the conditions to be satisfied by the chain to guarantee this behaviour
It’s All just Matrix Theory?

- consider $\pi_0$ is a vector of $N$ component defining, $\forall$ state $i$, the probability that the Markov chain is initially at $i$
  \[ \pi_0(i) = \mathbb{P}\{X_0 = i\}, \ i=1 \ldots N \]

- $\pi_n(j)$ defines the probability that the Markov chain is at $j$, after $n$ steps.
  \[ \pi_n = \{\pi_n(1), \ldots, \pi_n(N)\} \]

**Theorem**

Let $P$ be the transition matrix of a Markov chain, and let $\pi_0$ be the probability vector which represents the starting distribution. Then the probability that the chain is in state $s_i$ after $n$ steps is the $i$th entry of $\pi^n = \pi_0 P^n$
\[ \pi_n(j) \] defines, \( \forall \) state \( j \), the probability that the Markov chain is at \( j \), after \( n \) steps.

\[ \pi_n = \{ \pi_n(1), \ldots, \pi_n(N) \} \]

Thus, we have

\[ \pi_1 = \pi_0 P \]
\[ \pi_2 = \pi_1 P = \pi_0 P^2 \]
\[ \pi_3 = \pi_2 P = \pi_0 P^3 \]

In general

\[ \pi_n = \pi_0 P^n \]
It’s All Just Matrix Theory?

- let us suppose
  \[ \pi_0 = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \]
- we want to compute the state distribution after 3 frog jumps
  \[ \pi^3 = \pi_0 P^3 = \]
  \[
  \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \begin{pmatrix}
    0.258 & 0.295 & 0.442 \\
    0.244 & 0.404 & 0.342 \\
    0.244 & 0.392 & 0.355
  \end{pmatrix}
  \]
It’s All Just Matrix Theory?

No, not just matrix theory...

In principle, matrix algebra can give us the answer to any question about the probabilistic behaviour of a Markov chain, but this is not viable, in practice.

Consider a P2P system

- Markov chains describe random walk on P2P overlays
  - states corresponds to peer
  - transition correspond to link overlays
- the size of the matrix is huge, matrix algebra may be not exploited

The same for social graph analysis
A Random Walk on a Clock

- A simplified clock with 6 numbers: 0, 1, 2, 3, 4, 5
- From each state we can move clockwise, counter clockwise, or stay in place with the same probability.
- The transition matrix is:

\[
P(i, j) = \begin{cases} 
1/3 & \text{if } j = i-1 \mod 6 \\
1/3 & \text{if } j = i \\
1/3 & \text{if } j = i+1 \mod 6 
\end{cases}
\]
A Random Walk on a Clock

- Suppose we start out at $X_0 = 2$, that is
  $\pi_0 = (0, 0, 1, 0, 0, 0)$
- $\pi_1 = (0, 1/3, 1/3, 1/3, 0, 0)$
- $\pi_2 = (1/9, 2/9, 1/3, 2/9, 1/9, 0)$
- $\pi_3 = (3/27, 6/27, 7/27, 6/27, 3/27, 2/27)$
- Notice that the probability is spreading out away its initial concentration on the initial state 2
A Random Walk on a Clock

- now guess what is the state of the random walk at time 10000
- an intuitive answer: $X_{10000}$ is uniformly distributed over the 6 states
- this can be proven formally
  $$\pi_n \rightarrow \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right)$$
- after 10000 steps, the random walk has forgotten that it started out in state 2
Markov Chains: The Basic Theorem

Theorem (Basic Limit Theorem)

Any irreducible, aperiodic, Markov chain defined on a set of state $S$ and with a stochastic transition matrix $P$ has a unique stationary distribution $\pi$ with all its component positive. Furthermore, let $P^n$ be the $n$-th power of $P$, then

$$\lim_{n \to +\infty} P^n_{i,j} = \pi(j), \forall i, j \in S$$

to understand the theorem, we need to define the words

- stationary distribution
- aperiodic
- irreducible

We will explain these concepts by showing some examples where the limit does not exist
Stationary Distribution

- a distribution which satisfies
  \[ \pi = \pi P \]

- every finite state Markov chain has at least one stationary distribution

- A trivial example with infinitely many stationary distribution: when \( P \) is the identity matrix, in which case all distributions are stationary

- The stationary distribution is the one obtained by computing the power of the transition matrix
Markov Chains: Periodicity

- A state $i$ has period $k$ if any return of the chain to state $i$ occurs in multiples of $k$ time steps.
- State $i$ is periodic of period $k$, if $k > 1$, otherwise it is aperiodic.

- a periodic chain: clock like a Markov chain, each hour a state
- every state is visited every 12 hours with probability=1
- greatest common divisor of the return times is 12.
Markov Chains: Periodicity/Aperiodicity

- a Markov chain with 4 states, probability on edges $\neq 0$, (value not important).
- starting at state 2 you can return to 2 in 2 steps ($2 \rightarrow 1 \rightarrow 2$) or 3 steps ($2 \rightarrow 4 \rightarrow 1 \rightarrow 2$)
  - state 2 is aperiodic.
  - the greatest common divisor of the return times is 1
- if you remove the arrow from 2 to 4 all states would have period 2.
Markov Chain Periodicity

**Definition**

Given a Markov chain $X_0, X_1, \ldots$, the period $p_i$ of a state $i$ is the greatest common divisor (gcd)

$$p_i = \{ \text{gcd } n \text{ such that } P^n(i, i) > 0 \}$$

A state $i$ has period $k$ if any return to state $i$ must occur in multiples of $k$ time steps.

**Definition**

An irreducible Markov chain is aperiodic if its period is 1, and periodic otherwise.
Why periodicity is a problem for convergence?

Assume $P(X_0 = 1) = 1$. Then we have $a^{(0)} = [1, 0],

a^{(1)} = [0, 1],

a^{(2)} = [1, 0],

a^{(3)} = [0, 1],

\ldots

a^{(2n)} = [1, 0],

a^{(2n+1)} = [0, 1]

Conclusion: Limiting distribution does not exist!
Why periodicity is a problem for convergence?

Periodic chain with period 2

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad P^{2n} = \begin{pmatrix} 0.25 & 0 & 0 & 0.5 & 0.25 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0.25 & 0 & 0.5 & 0 & 0.25 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0.25 & 0 & 0.5 & 0 & 0.25 \end{pmatrix}$$

$$P^{2n+1} = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 & 0 \\ 0.25 & 0 & 0.5 & 0 & 0.25 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0.25 & 0 & 0.5 & 0 & 0.25 \\ 0 & 0.5 & 0 & 0.5 & 0 \end{pmatrix}$$

This means at every state $i$, if $P^n(i,i) > 0$, then $n$ is even.
Communicating States

- State \( j \) is reachable (or accessible) from state \( i \) if the probability to go from \( i \) to \( j \) in \( n \geq 0 \) steps, is greater than 0 (we assume every state is accessible from itself).

- Two states \( i \) and \( j \) are said to communicate \( i \leftrightarrow j \), if they are accessible from each other.

- Communication is an equivalence relation
  - every state communicates with itself \( i \leftrightarrow i \)
  - if \( i \leftrightarrow j \), then \( j \leftrightarrow i \);
  - if \( i \leftrightarrow j \) and \( j \leftrightarrow k \), then \( i \leftrightarrow k \).

- the states of a Markov chain can be partitioned into communicating classes such that only members of the same class communicate with each other. That is, two states \( i \) and \( j \) belong to the same class if and only if \( i \leftrightarrow j \).
Communicating Classes

- Consider following Markov chain where we assume that when there is an arrow from state $i$ to state $j$, then $p_{ij} > 0$
- What are the equivalence classes for this Markov chain?
Communicating Classes

- four communicating classes
  - Class 1 = \{state 1, state 2\}
  - Class 2 = \{state 3, state 4\}
  - Class 3 = \{state 5\}
  - Class 4 = \{state 6, state 7, state 8\}. 
State classification:

- **Recurrent states**: at any time the Markov chain enters Class $4=\{\text{state 6, state 7, state 8}\}$, it will always stay in that class.
- **Transient states**: Markov chain might stay in Class $1=\{\text{state 1, state 2}\}$ for a while, but at some point, it will leave that class and it never return to it.
Recurrent and Transient States

Diagram:

- States: 0, 1, 2, 3, 4
- Transient States: 1, 2
- Recurrent State: 0, 4
- Positive Recurrent States: 3
Markov Chain Irreducibility

- A Markov chain is said to be irreducible if all states communicate with each other.

- Irreducible Markov Chain

- Reducible Markov Chain

  Absorbing State

  Closed irreducible set
Why Reducibility is a problem for convergence?

Assume $P(X_0 = 1) = 1$

\[ a(0) = [1, 0, 0, 0] \]
\[ a(1) = [0, 0.5, 0.5, 0] \]
\[ a(2) = [0, 0, 0.5, 0.5] \]
\[ \ldots \]
\[ a(n) = [0, 0, 0.5, 0.5] \]

Assume $P(X_0 = 2) = 1$

\[ a(0) = [0, 1, 0, 0] \]
\[ a(1) = [0, 0, 0, 1] \]
\[ a(2) = [0, 0, 0, 1] \]
\[ \ldots \]
\[ a(n) = [0, 0, 0, 1] \]

Conclusion: Limiting distribution depends on initial distribution!
Reducible Chains

\[ P = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 \\
0 & 0 & \frac{3}{4} & \frac{1}{4} & 0 \\
0 & 0 & \frac{1}{8} & \frac{2}{3} & \frac{5}{24} \\
0 & 0 & 0 & \frac{1}{6} & \frac{1}{6}
\end{pmatrix} \]

\[ P^n = \begin{pmatrix}
0.25 & 0.75 & 0 & 0 & 0 \\
0.25 & 0.75 & 0 & 0 & 0 \\
0 & 0 & 0.182 & 0.364 & 0.455 \\
0 & 0 & 0.182 & 0.364 & 0.455 \\
0 & 0 & 0.182 & 0.364 & 0.455
\end{pmatrix} \]
A converging chain

Assume $P(X_0 = 1) = 1$

$\begin{align*}
\alpha^{(0)} &= [1.000, 0.000] \\
\alpha^{(1)} &= [0.500, 0.500] \\
\alpha^{(2)} &= [0.625, 0.375] \\
\alpha^{(3)} &= [0.594, 0.406] \\
\alpha^{(4)} &= [0.602, 0.398] \\
\alpha^{(5)} &= [0.600, 0.400] \\
\ldots
\end{align*}$

Assume $P(X_0 = 2) = 1$

$\begin{align*}
\alpha^{(0)} &= [0.000, 1.000] \\
\alpha^{(1)} &= [0.750, 0.250] \\
\alpha^{(2)} &= [0.563, 0.437] \\
\alpha^{(3)} &= [0.610, 0.390] \\
\alpha^{(4)} &= [0.598, 0.402] \\
\alpha^{(5)} &= [0.600, 0.400] \\
\ldots
\end{align*}$

Conclusion: Unique limiting distribution!
Analysis of DHT Routing by Markov Chains

- original papers on DHT provide claims on the asymptotic complexity of the routing
  - logarithmic for Chord, Kademlia
  - square root for CAN
- this does not give information on the \textit{exact distribution} of the routing length.
- our goal: modelling the routing process in order to know:
  - what is the percentage (probability) that routing requires exactly k steps?
- this can be done also by simulations, but....
  - high cost
  - difficult to cover the whole parameter space
- formal models require a few resources and are faster
Analysis of DHT Routing by Markov Chains

- knowing the exact distribution of routing lengths is relevant for:
  - setting time-outs
  - estimating message overhead

- *time-out*: repeat a query after a certain time when one can be relatively sure that the first query failed due to message loss
  - choose such a time-out as the 0.99 quantile of the routing length distribution

- message overhead: analysis of message traffic on the overlay
Markov Chains for Routing

- define a random process $X_0, X_1, X_2, \ldots$ where each $X_i$ describes a routing state.
  - each state will describe the distance from the current position of the key to the target

- define a final state $F$, *Found* corresponding to the successful finding of the key

- to get a routing distribution, follow the next steps:
  - define all possible routing states
  - derive initial distribution $I$ over all the states
  - derive the transition matrix $T$
  - compute $P_k = T^k I$ for all desired $k$, until routing is completed with probability 1
Markov Chains for Routing: A Simple Example

- a simple (non realistic, not Chord routing) scenario: each peer knows all other peers of the overlay
- a single target node storing the required key
- routing
  - two states $NF$ (Not Found), $F$ (Found)
  - asking one random node in a set of $n$ nodes
  - target is found with probability $1/n$ at each step
Markov Chains for Routing: A Simple Example

- **Markov Chain specification**
  - State Space: \( S = \{F, NF\} \), where \( F \) stands for having found the target and \( NF \) for not having found the target.
  - Initial probability distribution:
    \[
    I = (P(X_0 = F), P(X_0 = NF))^t = (0, 1)^t
    \]
  - Transition matrix \( T \):
    \[
    T = \begin{pmatrix}
    1 & 0 \\
    \frac{1}{n} & \frac{n-1}{n}
    \end{pmatrix}
    \]
Modelling Chord Routing by Markov Chains

- consider a Chord ring with identifier of $b$ bits, $2^b$ identifiers
- suppose $n$ nodes have joined the ring
- during the routing from a source $s$ to a destination $d$ (the node owning the content), the choice of the next hop depends on:
  - the distance from $s$ to $d$
  - a *random factor* introduced by the random selection of IDs made by the nodes, which is modelled as a *uniform distribution*. 
Modelling Chord Routing by Markov Chains

- **State space**: consists of all possible distances, $0, \ldots, 2^b - 1$ to the destination
  - a distance of 0 means that the destination has been reached
- **Random Variable** $X_i$: models the state of routing after $i$ routing steps. $X_0$ is the initial state of the routing.
- **The Initial Probability Distribution, $I$**:
  - the sender is one node whose identifier is chosen uniformly at random, excluding the identifier of $d$, the final destination.
  - hence, initially all non-zero distances from the destination are equally likely.

$$I = (P(X_0 = 0), P(X_0 = 1), \ldots, P(X_0 = 2^b - 1))^t = (0, \frac{1}{2^b-1}, \ldots \frac{1}{2^b-1})^t$$

- **The Transition Probability Matrix, $T$**: $t_{ji}$ is the probability of passing from distance $j$ to distance $i$ (see following slides)
The Transition Probability Matrix

• it remains to derive the transition probabilities

\[ t_{ji} = P(X_{k+1} = i|X_k = j) \]

the probability to go from distance \( j \) to distance \( i \)

• first of all, if the destination is already found, the routing remains in that state, so

\[ t_{i,0} = \begin{cases} 
1 & i = 0 \\
0 & \text{otherwise}
\end{cases} \]

• to determine the other entries of the transition matrix, keep in mind the Chord routing mechanisms!
Determining the Farthest Finger

- let us suppose to unroll the Chord ring: all the identifiers on a straight line
- the routing stops in node $Y$ which owns the key
- suppose that the routing has reached node $X$ whose distance to $Y$ is $j$, before next hop routing
- next hop: find the farthest finger $f$ that does not overtake $Y$
Determining the Farthest Finger

- first of all, find the maximum target (power of 2) which does not overtake Y that is to find
  \[ d = \max\{e : 2^e \leq j\} = \left\lfloor \log_2(j) \right\rfloor \]
- example:
  - \( j = 100, \left\lfloor \log_2(100) \right\rfloor = 6 \) gives the exponent of the maximum power of 2 that does not overtake 100
  - \( j = 28, \left\lfloor \log_2(28) \right\rfloor = 4 \) gives the exponent of the maximum power of 2 that \( 2^p \) does not overtake 28
Determining the Farthest Finger

- the farthest target is \( \text{target} = 2^\lfloor \log_2 j \rfloor \)
- \( d \) is the distance between the farthest target \( t \) and \( Y \)
  
  \[ d = j - 2^\lfloor \log_2 j \rfloor \]
First Scenario: Next Hop Brings to Destination

The next routing hop brings directly to $Y$

- no further node is located between $\text{target}$ and $Y$, so that
  - $Y$ is the finger corresponding to $\text{target}$
  - after next hop, the key is sent directly to $Y$ and the distance $i$ becomes 0
The next routing hop brings directly to $Y$: when does this happen?

- all $n - 2$ nodes (apart from $X$ and $Y$), selected an ID outside $d$ out of $2^b$ identifiers
- this happens with probability:

$$\left(\frac{2^b - d}{2^b}\right)^{n-2}$$
First Scenario: Next Hop Brings to Destination

- note that using a Markov Chain is an approximation, because knowing the number of nodes already on the path, the number of nodes that can be in the interval is slightly reduced.

- however, it can be shown that the difference is negligible.
Second Scenario: More Routings Hops Needed

The next routing hop brings to a node $Z$ closest to $Y$, but it does not bring directly to $Y$

- this happens if a node $Z \neq Y$ is located between target and the owner of the key $Y$
- the finger corresponding to target is $Z$
- the message is sent to $Z$
- the distance to $Y$, after the next hop, becomes $i \neq 0$
The next routing hop brings to a node Z closest to Y, but not to Y: when does this happen?

- no node between the target and Z, otherwise Z would not be the next hop.
- ...but Z must be between X and Y, so at least one node has chosen an identifier between that of X and that of Y.
The next routing hop brings to a node Z closest to Y, but not to Y: when does this happen?

- no node has selected an ID in an interval of length \( d - i \)
- at least one node has chosen an ID in an interval of length \( d - i + 1 \)
- this happens with probability

\[
\left( \frac{2^b - d + i}{2^b} \right)^{n-2} - \left( \frac{2^b - d + i - 1}{2^b} \right)^{n-2}
\]
An Impossible Scenario

the next hop cannot bring the message farther from the destination than the target

- the distance $i$ to $Y$, after the next hop cannot be larger than $d$
- this implies

$$\forall i > d, \text{ the probability is equal to 0}$$
Let $d = j - 2^{\lfloor \log_2 j \rfloor}$, the generic element of the transition matrix is the following one:

$$t_{ji} = \begin{cases} 
\left( \frac{2^b - d}{2^b} \right)^{n-2} & i = 0 \\
\left( \frac{2^b - d + i}{2^b} \right)^{n-2} - \left( \frac{2^b - d + i - 1}{2^b} \right)^{n-2} & 0 < i \leq d \\
0 & i > d 
\end{cases}$$
Computing Routing Length Distribution

- Let us come back to our original problem: compute the distribution of the lengths of the routing paths in Chord.
  - What is the percentage of routing hops ending after 1 hop? and of those ending after 2 hops? ....what about those ending after i hops?

- Compute the successive powers $T^k$ of the transition matrix.

- Take the initial distribution $I$, then compute $P_k = T^k I$.

- The i-th entry of the vector $P_k$ tells you the percentage of routings which have a distance $i$ from the target after $k$ routing steps.

- We are interested in the percentage of routings reaching the destination after $k$ steps. Look at the 0-th entry of $P_k$!

$$P_k(0) = P(X_k = 0)$$
An Idea for the Mid Term

- Write a JAVA program which takes in input $b$, the number of bits for the identifiers of a Chord ring, and $n$, and the number of nodes on the ring, and returns the length distribution for routing on that ring.

- Plot the resulting probability distribution. Compare the formal results with those obtained from a real implementation of Chord, for this you can download some open source implementation of Chord (for instance look at https://www.p2p.tu-darmstadt.de/research/gtna/ or at http://peersim.sourceforge.net)