Singular value decomposition

Two way to 'fix' the eigenvalue decomposition for a nonsymmetric matrix A:

- (Schur) $A = UTU^T$, with U orthogonal and T upper triangular.
- $A = U\Sigma V^T$, with U, V orthogonal and Σ diagonal.

Singular value decomposition (SVD)

For every $A \in \mathbb{R}^{m \times m}$, there are U, V orthogonal, Σ, diagonal s.t.

$$A = U\Sigma V^{-1} = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_m \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix}$$

$$= u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T + \cdots + u_m\sigma_m v_m^T.$$

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_m \geq 0$

For each square A there are U orthogonal, T upper triangular s.t.

$$A = UTU^T$$

$$\boxed{U}\,\boxed{T}\,\boxed{U}^T$$

$$A^k = VDV^{-1}VDV^{-1}\cdots VDV^{-1} = VD^kV^{-1} \quad \&$$

$$A^k = UTU^TUTU^T\cdots UTU^T = UT^kU^T$$

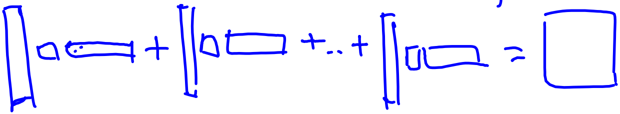
powers of triangular matrices are easier than the general case.

$$\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1m} \\ 0 & t_{22} & & \vdots \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & t_{mm} \end{bmatrix}^K = \begin{bmatrix} t_{11}^K & * & * \cdots & * \\ 0 & t_{22}^K & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & t_{mm}^K \end{bmatrix}$$

(diagonal elements = eigenvalues)

$$\begin{bmatrix} u_1 \,\Big|\, u_2 \,\Big|\, \cdots \,\Big|\, u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_m \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}^T =$$

$$= \begin{bmatrix} u_1\sigma_1 \,\Big|\, u_2\sigma_2 \,\Big|\, \cdots \,\Big|\, u_m\sigma_m \end{bmatrix} \begin{bmatrix} v_1^T \\ \hline v_2^T \\ \hline \vdots \\ \hline v_m^T \end{bmatrix} = u_1\sigma_1 v_1^T + \cdots + u_m\sigma_m v_m^T$$



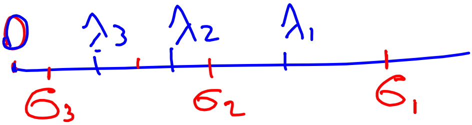
$$\begin{bmatrix} u_3 & u_2 & u_1 & u_4 \end{bmatrix} \begin{bmatrix} \sigma_3 & & & \\ & \sigma_2 & & \\ & & \sigma_1 & \\ & & & -\sigma_4 \end{bmatrix} \begin{bmatrix} v_3 & v_2 & v_1 & -v_4 \end{bmatrix}^T =$$

$$u_3 \sigma_3 v_3^T + \dots = A$$

Usually, one assumes

$$\boxed{\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_m \geq 0}$$

s.v.s are more "spread apart" than eigenvalues:



Singular value decomposition

linear (larger) quadratic (smaller)

`[U, S, V] = svd(A)` costs about $O(\boxed{mn^2})$ for $A \in \mathbb{R}^{m \times n}$ or $A \in \mathbb{R}^{n \times m}$ with $m \geq n$. (Need care about storing the $m \times m$ factor, though. More about it later.) Constant in front $\approx 20 - 30$.

The σ_i are called singular values and taken positive and ordered: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \geq 0$.

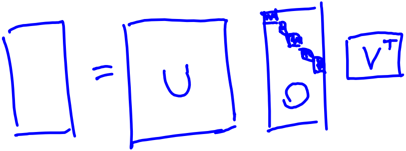
Singular values $\neq$ eigenvalues. They are always positive and usually more ‘spread apart’ than the eigenvalues. (Matlab examples)

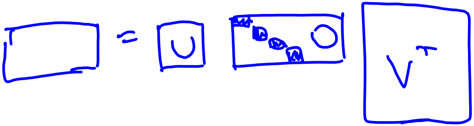
The SVD can be written also for a rectangular matrix $A \in \mathbb{R}^{m \times n}$. We have $U \in \mathbb{R}^{m \times m}$, $\Sigma \in \mathbb{R}^{m \times n}$ (padded with zeros), $V \in \mathbb{R}^{n \times n}$.

(Matlab examples)

$A \in \mathbb{R}^{m \times n}$ $\quad U \in \mathbb{R}^{m \times m}$ $\quad V \in \mathbb{R}^{n \times n}$

$\Sigma \in \mathbb{R}^{m \times n}$





Remark: if $A = U\Sigma V^T$, $A^T = V \Sigma^T U^T$

$$\begin{array}{cc} & n \\ m & \boxed{A} \end{array} = \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} \ddots \\ \hline 0 \end{bmatrix} \begin{bmatrix} & \end{bmatrix}^T = \begin{array}{c} \\ m \end{array}\begin{array}{cc} n & m-n \\ \left[\begin{array}{c|c} U_1 & U_2 \end{array}\right] \end{array} \begin{array}{c} \\ n \\ m-n \end{array}\begin{array}{c} n \\ \begin{bmatrix} S_1 \\ \hline 0 \end{bmatrix} \end{array} \begin{array}{c} n \\ \begin{bmatrix} V \end{bmatrix}^T_n \end{array} =$$

$$= \left(U_1 S_1 + \cancel{U_2 \, 0}\right) V^T = \underbrace{U_1}_{m \times n} \underbrace{S_1}_{n \times n} \underbrace{V^T}_{n \times n}$$

[Thin SVD]

Properties of SVD $\operatorname{rk} A = \dim\left(\{v : \underbrace{v = Aw \text{ for some } w}_{\operatorname{Im} A}\}\right)$

The SVD reveals rank, image, and kernel of a matrix.

Rank r = number of nonzero singular values:
$\sigma_1 \geq \cdots \geq \sigma_{\boxed{r}} > \sigma_{r+1} = \cdots = \sigma_n = 0.$
We can omit row/columns after r in the matrices:

$$A = U\Sigma V^{-1} = \begin{bmatrix} \\ u_1 & u_2 & \cdots & u_m \\ \\ \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix}$$

$$= \underline{u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T + \cdots + u_r\sigma_r v_r^T}.$$

For each $x \in \mathbb{R}^n$, Ax is linear combination of $u_1, \ldots, u_r$ (image).
Each of $v_{r+1}, \ldots, v_n$ (and their linear combinations) satisfy $Ay = 0$ (kernel).

$$Ax = \left(u_1 \sigma_1 v_1^T + \dots + u_r \sigma_r v_r^T\right) x =$$

$$= u_1 \underbrace{\sigma_1 v_1^T x} + \dots + u_r \underbrace{\sigma_r v_r^T} x$$

$=$

Exercises

1. If $A = U\Sigma V^T$ is the SVD of a square invertible A, what is the SVD of A^{-1}?
2. If A is positive semidefinite, is its eigendecomposition $A = U\Lambda U^T$ also an SVD?
3. If A is not positive semidefinite, how can we modify signs in $A = U\Lambda U^T$ to obtain an SVD?
4. Show that for a square $A = USV^T$ one has $AA^T = US^2U^T$ and $A^TA = V^TS^2V$, and that these are eigendecompositions.
5. How do the decompositions in the previous exercise change if A is rectangular? Check also with Matlab.