Singular value decomposition

Two way to 'fix' the eigenvalue decomposition for a nonsymmetric matrix A:

- (Schur) $A = UTU^T$, with U orthogonal and T upper triangular.
- $A = U\Sigma V^T$, with U, V orthogonal and Σ diagonal.

Singular value decomposition (SVD)

$$A = U\Sigma V^{-1} = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_m \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix}$$
$$= u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T + \cdots + u_m\sigma_m v_m^T.$$

Singular value decomposition

`[U, S, V] = svd(A)` costs about $O(mn^2)$ for $A \in \mathbb{R}^{m \times n}$ or $A \in \mathbb{R}^{n \times m}$ with $m \geq n$. (Need care about storing the $m \times m$ factor, though. More about it later.) Constant in front $\approx 20 - 30$.

The σ_i are called singular values and taken positive and ordered: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \geq 0$.

Singular values $\neq$ eigenvalues. They are always positive and usually more 'spread apart' than the eigenvalues. (Matlab examples)

The SVD can be written also for a rectangular matrix $A \in \mathbb{R}^{m \times n}$. We have $U \in \mathbb{R}^{m \times m}$, $\Sigma \in \mathbb{R}^{m \times n}$ (padded with zeros), $V \in \mathbb{R}^{n \times n}$.

(Matlab examples)

Properties of SVD

The SVD reveals rank, image, and kernel of a matrix.

Rank r = number of nonzero singular values:
$\sigma_1 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0$.
We can omit row/columns after r in the matrices:

$$A = U\Sigma V^{-1} = \begin{bmatrix} \\ u_1 & u_2 & \cdots & u_m \\ \\ \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix}$$

$$= u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T + \cdots + u_r\sigma_r v_r^T.$$

For each $x \in \mathbb{R}^n$, Ax is linear combination of $u_1, \ldots, u_r$ (image).
Each of $v_{r+1}, \ldots, v_n$ (and their linear combinations) satisfy $Ay = 0$ (kernel).

Exercises

1. If $A = U\Sigma V^T$ is the SVD of a square invertible A, what is the SVD of A^{-1}?
2. If A is positive semidefinite, is its eigendecomposition $A = U\Lambda U^T$ also an SVD?
3. If A is not positive semidefinite, how can we modify signs in $A = U\Lambda U^T$ to obtain an SVD?
4. Show that for a square $A = USV^T$ one has $AA^T = US^2U^T$ and $A^TA = V^TS^2V$, and that these are eigendecompositions.
5. How do the decompositions in the previous exercise change if A is rectangular? Check also with Matlab.