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A Comparison Result for Multisplittings Based on Overlapping Blocks and its Application to Waveform Relaxation Methods

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Seminar für Angewandte Mathematik
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Abstract

We show that certain multisplitting iterative methods based on overlapping blocks yield faster convergence than corresponding non-overlapping block iterations, provided the coefficient matrix is an M-matrix. This result is then applied to compare different waveform relaxation methods for solving initial value problems. Numerical experiments on the Intel iPSC/860 hypercube are included.

Keywords: multisplittings, overlapping blocks, comparison results, M-matrices, regular splittings, waveform relaxation, initial value problems, parallel algorithms

AMS(MOS) Subject Classification: 65F10, 65L05, 65W05

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1 Introduction

Consider the linear system

$$Ax = b$$

where $A \in \mathbf{R}^{n\times n}$ is regular. A *multisplitting* of A (see [9]) is a collection of matrices $M_l, N_l, E_l \in \mathbf{R}^{n\times n}$, $l = 1,\dots,L$, so that $A = M_l - N_l$, $l = 1,\dots,L$, where each M_l is regular and each E_l is nonnegative diagonal with $\sum_{l=1}^{L} E_l = I$. The iterative method belonging to this multisplitting calculates x^{n+1} from x^n via the L systems

$$M_l\, y^{n,l} = N_l\, x^n + b,\ l = 1,\dots,L$$

by setting

$$\begin{aligned} x^{n+1} = \sum_{l=1}^{L} E_l\, y^{n,l} &= \sum_{l=1}^{L} E_l M_l^{-1} N_l\, x^n + \sum_{l=1}^{L} E_l M_l^{-1} b \\ &=: Hx^n + c \quad . \end{aligned} \tag{1}$$

Multisplitting iterations are genuine parallel methods, since the $y^{n,l}$, $l = 1,\dots,L$ can be computed independently from each other. As was repeatedly observed in literature, (see [2, 3, 8, 9], e.g.) components of $y^{n,l}$ are not directly needed in (1) if the corresponding diagonal entry of E_l, the 'weighting', is zero. Hence, if the splitting $A = M_l - N_l$ allows so, these components need not be computed at all. In this paper however, we will show that it can be advantageous to compute certain components, although they will be weighted by zero.

To give a specific example let $L = 2$ and let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be a block decomposition of A with quadratic blocks A_{11}, A_{22}. Let D_{11}, D_{22} denote their respective diagonal parts. Assume that $A_{11}, A_{22}, D_{11}, D_{22}$ are all regular. Let I always denote the identity matrix of appropriate dimension. Consider the multisplitting

$$\left.\begin{array}{lll} M_1 = \begin{pmatrix} A_{11} & 0 \\ 0 & D_{22} \end{pmatrix} & , & M_2 = \begin{pmatrix} D_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \\[2ex] E_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} & , & E_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \\[2ex] N_1 = M_1 - A & , & N_2 = M_2 - A. \end{array}\right\} \tag{2}$$

Here the resulting multisplitting iteration (1) can also be rewritten in terms of the single splitting

$$A = M - N \text{ with } M = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$$

as

$$Mx^{n+1} = Nx^n + b \quad ,$$

and D_{11}, D_{22} are just dummy entries to make M_1, M_2 regular as formally required in the definition of a multisplitting.

Multisplittings allow a simple description of methods relying on an *overlapping* block-decomposition of A. In the example above, expand ing A_{11} and A_{22} will produce overlapping quadratic blocks $\tilde{A}_{11}, \tilde{A}_{22}$ with

$$A = \begin{pmatrix} \tilde{A}_{11} & * \\ * & \tilde{A}_{22} \end{pmatrix} .$$

The resulting multisplitting can now be written as

$$\left.\begin{array}{lll} \tilde{M}_1 = \begin{pmatrix} \tilde{A}_{11} & 0 \\ 0 & \tilde{D}_{22} \end{pmatrix} & , & \tilde{M}_2 = \begin{pmatrix} \tilde{D}_{11} & 0 \\ 0 & \tilde{A}_{22} \end{pmatrix}, \\[2ex] \tilde{E}_1 = \begin{pmatrix} I & & \\ & \tilde{E}_{11} & \\ & & 0 \end{pmatrix} & , & \tilde{E}_2 = \begin{pmatrix} 0 & & \\ & \tilde{E}_{22} & \\ & & I \end{pmatrix}, \\[2ex] \tilde{N}_1 = \tilde{M}_1 - A & , & \tilde{N}_2 = \tilde{M}_2 - A, \end{array}\right\} \tag{3}$$

where $\tilde{E}_{11}, \tilde{E}_{22}$ account for the overlapping par t with $\tilde{E}_{11} + \tilde{E}_{22} = I$. The dummy blocks $\tilde{D}_{11}, \tilde{D}_{22}$ again represent parts of the diagonal of A, for example. An obvious and often used choice for $\tilde{E}_1, \tilde{E}_2$ is setting all diagonal elements of $\tilde{E}_{11}$ and $\tilde{E}_{22}$ to $\frac{1}{2}$. In the overlapping part the components of $y^{n,1}$ and $y^{n,2}$ are then combined with equal weight.

Another possible choice is to take $\tilde{E}_1, \tilde{E}_2$ just as E_1, E_2 from (2). In this case we thus take a non-overlapping weighting, but the method is still different from the non-overlapping method, since $y^{n,1}$ and $y^{n,2}$ result from the expanded blocks.

Of course, the above example can immediately be generalized to $L > 2$ (and we will do so at the beginning of Section 2).

Several results in literature (see [1, 8]) give comparison results on the speed of convergence of certain multisplittings and the standard Jacobi or Gauss–Seidel iteration. However, up to now it has remained an open question whether overlapping schemes do really pay out, i.e. whether the spectral radius of the corresponding iteration matrix is less than that of the non-overlapping iteration. In the present paper we will show that this is indeed the case if A is an M-matrix and if the weighting is always done via the non-overlapping E_l from (2). We will then apply this result to also obtain a comparison theorem for certain waveform relaxation methods for solving ordinary differential equations. Finally, we include some numerical experiments on the Intel iPSC/860 hypercube.

2 Results

Our first definition generalizes the example given in the introduction to more than two blocks.

Definition 1 *Let $S_1, \ldots, S_L$ be a partition of $\{1, \ldots, n\}$, i.e. the S_l are pairwise disjoint nonempty subsets of $\{1, \ldots, n\}$, so that $\bigcup_{l=1}^{L} S_l = \{1, \ldots, n\}$. Moreover, l et $S_l \subseteq T_l \subseteq \{1, \ldots, n\}$ for $l = 1, \ldots, L$.*

a) The multisplitting (M_l, N_l, E_l), $l = 1, \ldots, L$ *of* $A \in \mathbf{R}^{n\times n}$ *where*

$$\left.\begin{array}{ll} M_l = (M_l)_{ij} & \text{with } (M_l)_{ij} = \left\{\begin{array}{ll} a_{ij} & \text{if } i \in S_l \text{ and } j \in S_l \\ a_{ii} & \text{if } i = j \\ 0 & \text{else} \end{array}\right. , \\ N_l = M_l - A, & \\ E_l = (E_l)_{ij} & \text{with } (E_l)_{ij} = \left\{\begin{array}{ll} 1 & \text{if } i = j \in S_l \\ 0 & \text{else} \end{array}\right. \end{array}\right\} \quad (4)$$

is termed a (non-overlapping) block Jacobi splitting of A*. Here, the iteration matrix* $H = \sum_{l=1}^{L} E_l M_l^{-1} N_l$ *also satisfies* $H = M^{-1}N$ *with* M, N *from the single splitting*

$$A = M - N,$$

where

$$M = (M)_{ij} \ \ \text{with } (M)_{ij} = \left\{\begin{array}{ll} a_{ij} & \text{if } i, j \in S_l \text{ for some } l \in \{1, \ldots, L\} \\ 0 & \text{else} \end{array}\right. \quad (5)$$

b) Any multisplitting $(\tilde{M}_l, \tilde{N}_l, \tilde{E}_l)$, $l = 1, \ldots, L$ *of* $A \in \mathbf{R}^{n\times n}$ *where*

$$\left.\begin{array}{ll} \tilde{M}_l = (\tilde{M}_l)_{ij} & \text{with } (\tilde{M}_l)_{ij} = \left\{\begin{array}{ll} a_{ij} & \text{if } i \in T_l \text{ and } j \in T_l \\ a_{ii} & \text{if } i = j \\ 0 & \text{else} \end{array}\right. \\ \tilde{N}_l = \tilde{M}_l - A, & \\ \tilde{E}_l = (\tilde{E}_l)_{ij} & \text{with } (\tilde{E}_l)_{ii} = 0 \text{ if } i \notin T_l \end{array}\right\} \quad (6)$$

is termed an (overlapping) block Jacobi multisplitting of A *(see [3]).*

For $A, B \in \mathbf{R}^{n\times n}$ we write $A \leq B$ if the corresponding inequality holds componentwise. According to [13], a splitting $A = M - N$ with $A, M, N \in \mathbf{R}^{n\times n}$ will be called *regular* or *weak regular* if $M^{-1} \geq 0$ and $N \geq 0$ or $M^{-1} \geq 0$ and $M^{-1}N \geq 0$, respectively.

We start with an important result on convergence of multisplitting iterations (see [1, 2, 9]).

Theorem 1 *Let* $(M_l, N_l, E_l), l = 1, \ldots, L$ *be a multisplitting of* A *and* $A^{-1} \geq 0$*. Assume that* $A = M_l - N_l$ *is a weak regular splitting for* $l = 1, \ldots, L$*. Denote* $H = \sum_{l=1}^{L} E_l M_l^{-1} N_l$*. Then* *(i)* $\rho(H) < 1$,

(ii) $\sum_{l=1}^{L} E_l M_l^{-1}$ *is non-singular and* $H = \tilde{M}^{-1}\tilde{N}$ *where*

$$\tilde{M} = \left(\sum_{l=1}^{L} E_l M_l^{-1} \right)^{-1}, \ \tilde{N} = \tilde{M} - A, \tag{7}$$

(iii) $A = \tilde{M} - \tilde{N}$ *is a weak regular splitting.*

Proof: (i) was already shown in [9]. Since $N_l = M_l - A$ we have

$$H = I - \left(\sum_{l=1}^{L} E_l M_l^{-1} \right) A \, , \tag{8}$$

and thus $\rho(H) < 1$ implies that $\sum_{l=1}^{L} E_l M_l^{-1}$ is non-singula r. In addition, (8) immediately shows $H = \tilde{M}^{-1}\tilde{N}$, thus proving (ii). Finally, since $M_l^{-1} \geq 0$ for $l = 1, \ldots, L$ we have $\tilde{M}^{-1} = \sum_{l=1}^{L} E_l M_l^{-1} \geq 0$ and $\tilde{M}^{-1}\tilde{N} = H = \sum_{l=1}^{L} E_l M_l^{-1} N_l \geq 0$. So $A = \tilde{M} - \tilde{N}$ is a weak regular splitting. □

In order to compare different multisplitting iterations, it would be useful to have comparison results for weak regular splittings. Our basis is the following lemma due to Elsner [1].

Lemma 1 *Let* $A \in \mathbf{R}^{n \times n}$ *be nonsingular with* $A^{-1} \geq 0$ *and assume that* $A = M - N = \tilde{M} - \tilde{N}$ *are two splitting s of* A *so that*

$M^{-1} \geq 0$, $N \geq 0$, (*i.e.* $M - N$ *is a regular splitting of* A)
$\tilde{M}^{-1} \geq 0$, $\tilde{M}^{-1}\tilde{N} \geq 0$ (*i.e.* $\tilde{M} - \tilde{N}$ *is a weak regular splitting of* A)

and

$$M^{-1} \leq \tilde{M}^{-1}.$$

Then

$$\rho(\tilde{M}^{-1}\tilde{N}) \leq \rho(M^{-1}N).$$

This lemma shows that a well-known comparison result for regular splittings (see [13]) also holds if one of the splittings is only weak regular. However, as was shown in [1], too, it does no longer hold if *both* splittings are only weak regular.

Together with Theorem 1 we obtain the following corollary of Lemma 1.

Corollary 1 *Let* (M_l, N_l, E_l), $l = 1, \ldots, L$ *and* $(\tilde{M}_l, \tilde{N}_l, E_l)$, $l = 1, \ldots, L$ *be two multisplittings of* A *using the same weighting matrices* E_l. *Assume that each splitting* $M_l - N_l$ *and* $\tilde{M}_l - \tilde{N}_l$, $l = 1, \ldots, L$ *is weak regular. Moreover, suppose that the splitting* $A = M - N$ *with* $M = (\sum_{l=1}^{L} E_l M_l^{-1})^{-1}$ *is regular.*

Then the corresponding multisplitting iteration matrices H *and* $\tilde{H}$ *satisfy*

$$\rho(\tilde{H}) \leq \rho(H),$$

provided

$$M_l^{-1} \leq \tilde{M}_l^{-1},\ l = 1, \ldots, L. \tag{9}$$

Proof : By Theorem 1, the matrix $\tilde{H}$ results from a weak regular splitting $A = \tilde{M} - \tilde{N}$ with $\tilde{M}^{-1} = \sum_{l=1}^{L} E_l \tilde{M}_l^{-1}$. So the Corollary follows from Lemma 1 once we have shown $M^{-1} \leq \tilde{M}^{-1}$. But this a direct consequence of (9), the weighting matrices E_l being equal in both multisplittings. □

We now want to apply this corollary to block Jacobi multisplittings of an M-matrix. Recall that $A = (a_{ij}) \in \mathbf{R}^{n \times n}$ is called an *M-Matrix*, if $a_{ij} \leq 0$ for $i \neq j$ and A is regular with $A^{-1} \geq 0$.

We need the following well-known result for M-matrices [10].

Lemma 2 *Let* $A = (a_{ij}) \in \mathbf{R}^{n \times n}$ *be an M-matrix and denote its diagonal part by* $D = diag(a_{11}, \ldots, a_{nn})$. *Then*

(i) $a_{ii} > 0$ *for* $i = 1, \ldots, n$. *In particular,* D *is regular .*

(ii) Any $B \in \mathbf{R}^{n \times n}$ *with* $A \leq B \leq D$ *is an M-matrix .*

(iii) If $C \in \mathbf{R}^{n \times n}$ *is another M-Matrix with* $A \leq C$ *then* $C^{-1} \leq A^{-1}$.

We are now able to state our central theorem. As opposed to the results presented so far, we now have to assume that the weighting is done through *non-overlapping* matrices E_l.

Theorem 2 *Let* $A = (a_{ij}) \in \mathbf{R}^{n \times n}$ *be an M-matrix. Let* $S_1, \ldots, S_L, T_1, \ldots, T_L \subseteq \{1, \ldots, n\}$ *be th e same as in Definition 1 and let* (M_l, N_l, E_l) *denote the corresponding non-overlapping block Jacobi splitting (4) and* $(\tilde{M}_l, \tilde{N}_l, E_l)$ *the overlapping block Jacobi multisplitting (6) using the same weighting matrices* E_l. *Then we have*

$$\rho(\tilde{H}) \leq \rho(H), \tag{10}$$

where $H = \sum_{l=1}^{L} E_l M_l^{-1} N_l$, $\tilde{H} = \sum_{l=1}^{L} E_l \tilde{M}_l^{-1} \tilde{N}_l$.

Proof: For $l = 1, \ldots, L$ we have

$$A \le \tilde{M}_l \le M_l \le D := \mathrm{diag}(a_{11}, \ldots, a_{nn}).$$

Hence, by Lemma 2

$$0 \le M_l^{-1} \le \tilde{M}_l^{-1},\ l = 1, \ldots, L.$$

Moreover, since the off-diagonal entries of A are non-positive, we get

$$N_l \ge 0,\ \tilde{N}_l \ge 0,\ l = 1, \ldots, L.$$

So each of the splittings $M_l - N_l$, $\tilde{M}_l - \tilde{N}_l$ is weak regular (and even regular). In addition, $M = (\sum_{l=1}^{L} E_l M_l^{-1})^{-1}$ is given by (5). The same argument as above shows that the splitting $A = M - N$ is again regular. The assertion of the theorem now follows directly from Corollary 1. □

Remark: Instead of block Jacobi multisplittings we can also consider block 'Gauss–Seidel–type' multisplittings. The matrices $M_l, \tilde{M}_l$ are then given by the lower triangular parts of the original matrices $M_l, \tilde{M}_l$ in (4) and (6).

In this case, the overlapping scheme is again superior to the non-overlapping scheme, i.e. (10) holds as well. This result follows exactly in the same manner as Theorem 2.

3 Results for ODEs

In 1982, Lelarasmee (see [6]) introduced a splitting technique similar to the block Jacobi splittings of Definition 1 for solving ordinary differential equations. In its original formulation this so-called *waveform relaxation algorithm* is restricted to non-overlapping subsystems, whereas the more general *multisplitting waveform relaxation algorithm*, introduced in [11] allows the subsystems to overlap.

To describe the multisplitting waveform relaxation algorithm in detail, consider a linear initial value problem of dimension n:

$$x'(t) + Ax(t) = f(t), \quad x(0) = x_0 \tag{11}$$

where $t \in [0, T]$, $x \in C^1([0, T]; \mathbf{R}^n)$, $x_0 \in \mathbf{R}^n$, f piecewise continuous , $A \in \mathbf{R}^{n \times n}$.

In an analogous way to (1) the initial value problem (11) is now solved iteratively via a multisplitting (M_l, N_l, E_l), $l = 1, \ldots, L$ of the matrix A. One thus solves the L *subsystems*

$$\left.\begin{array}{rcl} y'_{l,n+1}(t) + M_l y_{l,n+1}(t) & = & N_l x_n(t) + f(t), \\ y_{l,n+1}(0) & = & x_0. \end{array}\right\} \tag{12}$$

After the L subsystems have been solved, one obtains a new approximation to the solution by

$$x_{n+1}(t) = \sum_{l=1}^{L} E_l y_{l,n+1}(t). \tag{13}$$

As in the linear systems case, the solution of the L subsystems can be comput ed in parallel. The subsystems are solved not for just one timepoint, but for a whole time interval $[0, T]$. The iteration is performed until a certain stopping criterion, e.g. $\|x_{n+1} - x_n\| < \epsilon$ is satisfied for some appropriate norm $\|\cdot\|$ in $C^1([0, T]; \mathbf{R}^n)$.

The waveform relaxation algorithm has been studied in various papers. A collection of results for nonlinear problems that arise in the simulation of electrical circuits can be found in [14]. A much more detailed mathematical analysis for linear initial value problems was done by Nevanlinna in [7]. The convergence of the multisplitting waveform relaxation algorithm for linear problems is studied in [5] and [11].

In practical computations the subsystems have to be solved numerically, i.e. one calculates approximations to $y_{l,n+1}(t)$ at a finite number of time points in $[0, T]$. If one uses r equidistant timepoints with distance h, it can be shown that the iteration of the resulting multisplitting waveform relaxation algorithm is equivalent to a linear fixpoint iteration in $\mathbf{R}^{rn}$ [12]. As was also shown in [12] the spectral radius of the iteration matrix is given by

$$\rho(H) = \rho(\sum_{l=1}^{L} E_l (I + hM_l)^{-1} hN_l). \tag{14}$$

if the *implicit Euler–method* is used for discretizing the subsystems. (Note that H is $n \times n$ in (14)). Convergence of the implicit Euler multisplitting waveform relaxation algorithm is thus ensured for arbitrary A and arbitrary splittings if the stepsize h is sufficiently sma ll. Moreover, convergence is also guaranteed for arbitrary h if A is an M-Matrix and (M_l, N_l, E_l) is an (overlapping) Jacobi multisplitting or a (nonoverlapping) block Jacobi splitting (see [12]).

Given the results of the previous section, we can now state an analogous comparison result for multisplitting waveform relaxation algorithms. It shows that with the implicit Euler-method, a multisplitting waveform relaxation algorithm using an overlapping block Jacobi splitting will converge faster than without overlapping, provided the weighting is done with the matrices E_l from (4). More precisely, we have the following theorem.

Theorem 3 *Let $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ be an M-matrix. Let $S_1, \ldots, S_L, T_1, \ldots, T_L \subseteq \{1, \ldots, n\}$ be th e same as in Definition 1 and let (M_l, N_l, E_l) denote the corresponding non-overlapping block Jacobi splitting (4) and $(\tilde{M}_l, \tilde{N}_l, E_l)$ the overlapping block Jacobi multisplitting (6) using the same weighting matrices E_l. Then we have*

$$\rho(\tilde{H}) \leq \rho(H), \tag{15}$$

where $H = \sum_{l=1}^{L} E_l(I + hM_l)^{-1}hN_l$, $\tilde{H} = \sum_{l=1}^{L} E_l(I + h\tilde{M}_l)^{-1}h\tilde{N}_l$.

Proof: The proof is a simple application of Theorem 2, with A replaced by $I + hA$, $\tilde{M}_l$ by $I + h\tilde{M}_l$, M_l by $I + hM_l$, D by $I + hD$, N_l by hN_l and $\tilde{N}_l$ by $h\tilde{N}_l$. □

Theorem 3 again remains true if we use 'Gauss-Seidel-type' multisplittings instead of block Jacobi multisplittings.

Moreover, Theorem 3 is also valid for other discretization techniques. For example, if the trapezoidal rule is used for discretizing the subsystems (12), the iteration matrix has the form:

$$H = \sum_{l=1}^{L} E_l(I + \frac{h}{2}M_l)^{-1}\frac{h}{2}N_l.$$

So here the comparison result follows in exactly the same manner as Theorem 3.

If an implicit linear multistep method of the form

$$\sum_{i=0}^{k} \alpha_i y_{l,n+1,m+i} = h\sum_{i=0}^{k} \beta_i(-M_l y_{l,n+1,m+i} + N_l x_{n,m+i} + f_{m+i})$$

is used, the iteration matrix is given by

$$H = \sum_{l=1}^{L} E_l(\alpha_k I + h\beta_k M_l)^{-1} h\beta_k N_l.$$

Hence, the comparison result follows again easily if both, α_k and β_k, are positive. So here we have to require $A(\alpha)$ stability of the linear multistep method.

4 Numerical Results

In this section we will report on two experiments with multisplitting waveform relaxation algorithms performed on 16 processors of an Intel iPSC/860 hypercube. Our first example considers the semi discretized two–dimensional heat equation

$$u_t = u_{xx} + u_{yy}, \quad u(0, x, y) \equiv 0$$

on the unit square, with boundary conditions uniformly set to 1. The space discretization is done by using a five–point stencil using a stepsize $k = \frac{1}{16}$ in both directions. This results in a $n = 225$ dimensional linear system of ordinary differential equations of the form (11) with a block tridiagonal matrix

$$A = \begin{pmatrix} B & -I & & & \\ -I & B & -I & & \\ & \ddots & \ddots & \ddots & \\ & & & & -I \\ & & & -I & B \end{pmatrix} \quad \text{with } B = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 4 \end{pmatrix},$$

where $B, I \in \mathbf{R}^{15\times 15}$. It is well known that this matrix A is an M-matrix. The function $f(t)$ on the right hand side of (11) contains the boundary conditions.

We now define $S_l = \{15(l-1)+1, \ldots, 15l\}$, $l = 1, \ldots, 15$. So the resulting (non-overlapping) block Jacobi splitting from Definition 1 a) is induced by the single splitting $A = M - N$ where M is block diagonal with 15 diagonal blocks, each equal to B. Given a parameter *overlap* $\in \mathbf{N}$ we enlarge each index set S_l by appending *overlap* additional elements, i.e.

$$\begin{aligned} T_l &= \{15(l-1)+1,\ldots,15l,\ldots,15l+\mathit{overlap}\}, \\ &\quad l=1,\ldots,14, \\ T_{15} &= S_{15}. \end{aligned}$$

In this manner we define an overlapping block Jacobi multisplitting according to Definition 1 b). The weighting matrices were always chosen to be the non-overlapping E_l from (4). Table 1 reports the timings and the number of iterations for the corresponding multisplitting waveform relaxation algorithm(12), (13). The integration was performed using the implicit Euler–method over $t \in [0,1]$ with the starting function $u_0(t,x,y) = u_0 = 0$ and equidistant step sizes $h = 0.05$. Our stopping criterion was to check whether $\|x_{n+1} - x_n\|_\infty < 10^{-2}$ for all discrete time points. We used 15 processors of the iPSC/860, the 16th processor performing I/O and other auxiliary tasks.

In accordance with our theoretical results, Table 1 shows that the number of iterations with overlap is less than without overlap. This is also true for the respective execution times although increasing the overlap me ans increasing the amount of work to be done on each individual system. Table 1 thus shows that this additional work is more than compensated by the decreasing number of iterations. With *overlap* $= 30$ the time taken is about a third of the time taken for the case without overlap.

The results of Table 1 and of other numerical experiments suggest the conjecture, that the spectral radius of the iteration matrix decreases as the overlap increases. We were not able to show this theoretically, since we then would have to compare two weak regular splittings, for which Lemma 1 is no longer valid. The following example shows that the above conjecture is very likely to be wrong, as soon as one deviates from the *nonoverlapping* weighting scheme, i.e. if one chooses matrices E_l different from (4).

So, let again $n = 225$ and $L = 15$ and consider $A \in \mathbf{R}^{n\times n}$ with

$$a_{ij} = \begin{cases} 2 & \text{if } i=j \\ -(2^{|i-j|}) & \text{if } i \neq j \text{ and } |i-j| \leq 5 \\ -\frac{1}{64} & \text{if } (i,j) \in \{(1,n),(n,1)\} \\ 0 & \text{else} \end{cases} .$$

A is strictly diagonally dominant and thus, given its sign pattern, an M-matrix. The right hand side $f(t)$ and the initial value in (11) were both

overlap	iterations	time (ms)	% time	% iterations
0	232	31 004	100.00	100.00
1	231	30 846	99.49	99.57
2	230	30 767	99.24	99.14
3	226	30 457	98.24	97.41
4	221	29 856	96.30	95.26
5	214	29 196	96.49	92.24
6	206	28 282	91.22	88.79
7	196	27 147	87.56	84.48
8	184	25 694	82.87	79.31
9	172	24 290	78.34	74.14
10	159	22 639	73.02	68.53
11	147	21 194	68.36	63.36
12	137	19 925	64.27	59.05
13	130	19 136	61.72	56.03
14	126	18 656	60.17	54.31
15	125	18 633	60.10	53.88
16	123	18 363	59.23	53.02
17	121	18 133	58.49	52.16
18	118	17 668	56.99	50.86
19	114	17 291	55.77	49.14
20	109	16 594	53.52	46.98
21	105	16 175	52.17	45.26
22	100	15 465	49.88	43.10
23	95	14 912	48.10	40.95
24	88	14 007	45.18	37.93
25	82	13 202	42.58	35.34
26	76	12 482	40.26	32.76
27	71	11 827	38.15	28.88
28	67	11 317	36.50	28.88
29	66	11 266	36.34	28.45
30	66	11 267	36.34	28.45

Table 1: Results for the heat equation

taken to be discretizations of the sine function, but these choices do not significantly affect the results to be reported. The sets S_l and T_l were taken identical to our previous example. Hence, if $1 \le \mathit{overlap} \le 15$ we end up with areas of overlap between two consecutive blocks only. Table 2 now reports the number of iterations necessary to satisfy our stopping criterion for two different choices of weighting matrices. 'Non-overlapping' refers to the E_l defined in (4), whereas 'equiweighting' describes the scheme where we take the arithmetic mean whenever there are two contributions for one component. Formally, this means

$$(E_1)_{ii} = \begin{cases} 0 & \text{if } i \notin T_1 \\ 1 & \text{if } i \le 15 \\ \frac{1}{2} & \text{if } 15 < i \le 15 + \mathit{overlap} \end{cases} ,$$

$$(E_l)_{ii} = \begin{cases} 0 & \text{if } i \notin T_l \\ \frac{1}{2} & \text{if } 15(l-1) < i \le 15(l-1) + \mathit{overlap} \\ \frac{1}{2} & \text{if } 15l < i \le 15l + \mathit{overlap} \\ 1 & \text{if } 15(l-1) + \mathit{overlap} < i \le 15l \end{cases} \qquad \text{for } 1 < l < L ,$$

$$(E_L)_{ii} = \begin{cases} 0 & \text{if } i \notin T_L \\ \frac{1}{2} & \text{if } 15(L-1) < i \le 15(L-1) + \mathit{overlap} \\ 1 & \text{if } 15(L-1) + \mathit{overlap} < i \end{cases} .$$

The integration was performed over $t \in [0, 1]$ using stepsizes of $h = \frac{1}{20}$ with the implicit Euler–method.

overlap	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
equiweighting	12	10	9	8	7	6	6	6	5	5	5	7	8	9	11	13
nonoverlapping	12	10	9	8	7	6	6	5	5	5	4	4	4	4	4	4

Table 2: Equiweighting and nonoverlapping weighting

The last row of Table 2 again confirms our theoretical results. The equiweighting case is never better than the nonoverlapping case, a fact that we also noticed in other computations. Moreover, with equiweighting the number of iterations starts to increase with the overlap once $\mathit{overlap} \ge 11$. For $\mathit{overlap} = 15$ we even need one iteration more than without overlap. However, this last observation seems not significant enough to draw conclusions on the spectral radii of the corresponding iteration matrices.

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