# Pseudoinverse

The solution of a least-squares problem $\min \|Ax - b\|$ is given by

$$x = \boxed{(A^T A)^{-1} A^T} b$$

(if $A$ has full column rank).

$A^+ := (A^T A)^{-1} A^T$

## Definition

The (Moore-Penrose) pseudoinverse of a matrix $A$ with full column rank is $A^+ := (A^T A)^{-1} A^T$.

So we can write $x = A^+ b$ for the solution of a LS problem.

This Generalizes the concept of inverse — exists also for non-square $A$.

Not trivial: solution obtained by multiplying $b$ by a certain matrix. (In particular, the solution of $\min \|Ax - (b_1 + b_2)\|$ is the sum of the two solutions of $\min \|Ax_1 - b_1\|$ and $\min \|Ax_2 - b_2\|$ ).

Note that $A^+ A = I_n$, but $AA^+ \neq I_m$ (there can be no matrix such that $AA^+ = I_m$, for rank reasons.)

$$\min_{x\in\mathbb{R}^n} \|Ax-b\| \quad \rightsquigarrow \quad x = (A^TA)^{-1}A^Tb$$

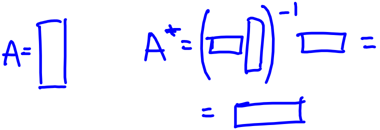
$$\min_{y\in\mathbb{R}^n} \|Ay-c\| \quad y = (A^TA)^{-1}A^Tc$$

$$\min_{z\in\mathbb{R}^n} \|Az-(b+c)\| \quad z = x+y = (A^TA)^{-1}A^T(b+c)$$

not obvious

$$\min \|Ax-b\| \quad \rightsquigarrow \quad x = A^+b$$

$$Ax = b \qquad x = A^{-1}b$$



$$A^+ = (A^TA)^{-1}A^T$$

$$\underset{n\times m}{A^+}\cdot\underset{m\times n}{A} = (A^TA)^{-1}(A^TA) = I_{n\times n}$$

$$A\cdot A^+ = A(A^TA)^{-1}A^T \underset{\text{in general}}{\neq} I_{m\times m}$$

If $m>n$, then this <u>can't</u> be the identity

$$\operatorname{Im} A(A^TA)^{-1}A^T \subseteq \operatorname{Im} A$$

$$A\underbrace{(A^TA)^{-1}Av} \qquad\qquad Aw$$

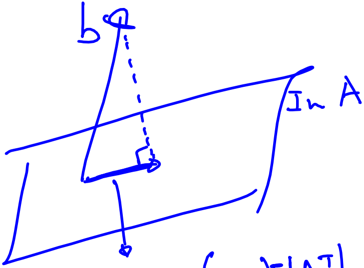
$\dim \operatorname{Im} A(A^TA)^{-1}A^T \leq \dim \operatorname{Im} A \leq n$

$\dim \operatorname{Im} I = m$

$$A(A^TA)^{-1}A^T \neq I$$

Remark:

$A(A^TA)^{-1}A^T$ is the orthogonal projector on $\operatorname{Im} A$




$Ax = A(A^TA)^{-1}A^Tb$