Stability and residual

Suppose we have solved a linear system...

```
>> A = randn(4, 4); b = randn(4, 1);
>> x = A \ b;
>> A*x - b
ans =
           0
  -1.3878e-17
           0
   2.2204e-16
```

To check if a computed solution $\widetilde{x}$ is accurate, usually one checks that $A\widetilde{x} - b = r$ ('residual') has small norm.

But does it imply that $\widetilde{x}$ is close to the true solution x?

Resiudal and error

Yes, because $\widetilde{x}$ is the exact solution of

$$A\widetilde{x} = b + r = b + \Delta b$$

and we know that in this case

$$\frac{\|x - \widetilde{x}\|}{\|x\|} \leq \kappa(A) \frac{\|r\|}{\|b\|}.$$

(Note that the computed $r = A\widetilde{x} - b$ might be inaccurate, too, but we are ignoring this issue now.)

Residual test for least squares problems

Now let's turn to least squares problems instead: $\min\|Ax - b\|$, with computed solution $\widetilde{x}$.

Can one expect $r = A\widetilde{x} - b$ to have small norm? No: it's the 'distance' between b and image(A), which can be large (depends on the problem).

What's small then? AF would say 'the derivative: it's a minimum'.

$$A^T A\widetilde{x} - A^T b.$$

Well, kind-of. Even if $\widetilde{x} = x + \Delta x$, one has only

$$\frac{\|A^T A\widetilde{x} - A^T b\|}{\|A^T b\|} \leq \kappa(A^T A)\frac{\|\Delta x\|}{\|x\|}.$$

and, as we know, $\kappa(A^T A) = \kappa(A)^2$ may be a lot larger than $\kappa(A)$.

Another residual test for LS problems

Let $A = Q_1 R_1$ be a thin QR factorization. Let $r_1 = Q_1^T(A\widetilde{x} - b)$. Then, $\widetilde{x}$ is the exact solution of the LS problem

$$\min\|Ax - (b + Q_1 r_1)\|, \quad \text{and } \|Q_1 r_1\| = \|r_1\|.$$

Proof (sketch): replay the solution of a LS problem with QR factorization, and use $Q_1^T Q_1 = I$. You will get in the first block $R_1 x = Q_1^T b + r_1$, i.e., $Q_1^T(Ax - b) = r_1$, which is verified by $\widetilde{x}$.

(Note that, in practice, we won't have access to the exact value of $Q_1^T(A\widetilde{x} - b)$), but still this bound can give us an indication of the accuracy of the computed solution.)