Conditioning of least squares problems

Conditioning of linear least squares is a more complicated problem than the one for linear systems.
We won't give a full proof:

Theorem (Trefethen, Bau, Theorem 18.1)

Consider the linear least squares problem $\min\|Ax - b\|$, with $A \in \mathbb{R}^{m\times n}$ with full column rank. Its relative condition number with respect to the input b is

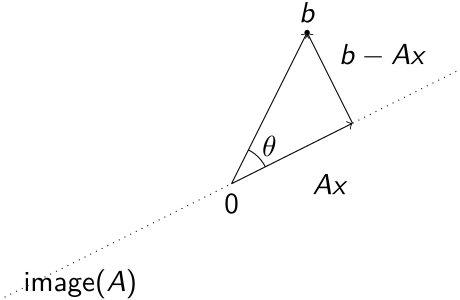
$$\kappa_{rel,b\to x} \leq \frac{\kappa(A)}{\cos\theta},$$

and with respect to A it is

$$\kappa_{rel,A\to x} \leq \kappa(A) + \kappa(A)^2 \tan\theta,$$

where θ is the angle such that $\cos\theta = \frac{\|Ax\|}{\|b\|}$.

The geometric picture



b 'split' into two orthogonal components: Ax and $b - Ax$.
QR and SVD reveal their norms: if $A = QR, Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ or $A = U\Sigma V^T$, $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ (as in their thing versions) then

$$\|Ax\| = \|Q_1^T b\| = \|U_1^T b\| = \|b\| \cos\theta,$$
$$\|b - Ax\| = \|Q_2^T b\| = \|U_2^T b\| = \|b\| \sin\theta.$$

Some intuition

A small $\frac{\|\Delta b\|}{\|b\|}$ may correspond to a large $\frac{\|Q_1^T \Delta b\|}{\|Q_1^T b\|}$ — the denominator differs by a factor $\cos\theta$.

Solving a linear system + this possible error amplification gives $\kappa_{rel,b\to x} \leq \frac{\kappa(A)}{\cos\theta}$.

This perturbation is more substantial if b and Ax are almost orthogonal, i.e., $\cos\theta \approx 0$.

Some intuition

Perturbing ΔA may 'tilt' image(A): this changes both $Q_1^T b$ and the matrix in the linear system.

This perturbation is less substantial if Ax and b are almost aligned, i.e., $\sin\theta \approx 0$.