Markov Random Fields

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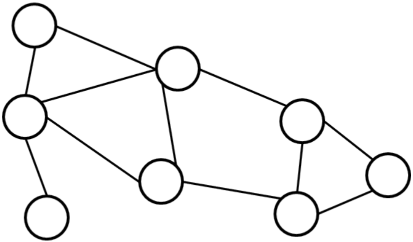
Intelligent Systems for Pattern Recognition (ISPR)

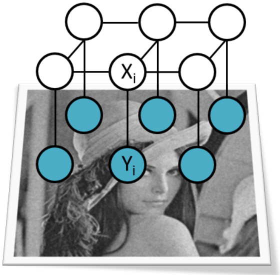
Markov Random Fields (MFRs)



- Undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ (a.k.a. Markov Networks)
- Nodes $v \in \mathcal{V}$ represent random variables X_v
  - Shaded $\Rightarrow$ observed
  - Empty $\Rightarrow$ un-observed
- Edges $e \in \mathcal{E}$ describe bi-directional dependencies between variables (constraints)

Graph often coherent with data structure

MRF Applications



Machine vision uses MRF to impose smoothness constraints on neighboring pixels

- Image denoising
  - Lattice Markov Network (Ising model)
  - $Y_i \rightarrow$ observed value of the noisy pixel
  - $X_i \rightarrow$ unknown (unobserved) noise-free pixel value
- Complexity of (exact) inference and learning can be impossible/impractical for complex graph structures

Likelihood Factorization

Define $\mathbf{X} = X_1, \ldots, X_N$ as the RVs associated to the N nodes in the undirected graph $\mathcal{G}$

$$P(\mathbf{X}) = \frac{1}{Z} \prod_{C} \psi_C(\mathbf{X}_C)$$

- $\mathbf{X}_C \rightarrow$ RV associated with nodes in the maximal clique C
- $\psi_C(\mathbf{X}_C) \rightarrow$ potential function for clique C
- $Z \rightarrow$ partition function ensuring normalization

$$Z = \sum_{\mathbf{X}} \prod_{C} \psi_C(\mathbf{X}_C)$$

Potential Functions

- Potential functions $\psi_C(\mathbf{X}_C)$ are not probabilities!
- Express which configurations of the local variables are preferred
- If we restrict to strictly positive potential functions, the Hammersley-Clifford theorem provides guarantees on the distribution that can be represented by the clique factorization

Definition (Boltzmann distribution)

A convenient and widely used strictly positive representation of the potential functions is

$$\psi_C(\mathbf{X}_C) = \exp\{-E(\mathbf{X}_C)\}$$

where $E(\mathbf{X}_C)$ is called energy function

Factorizing Potential Functions

In general, we will assume to work with MRF where the partition functions factorize as

$$\psi_C(\mathbf{X}_C) = \exp\left(\sum_k \theta_{Ck} f_{Ck}(\mathbf{X}_C)\right)$$

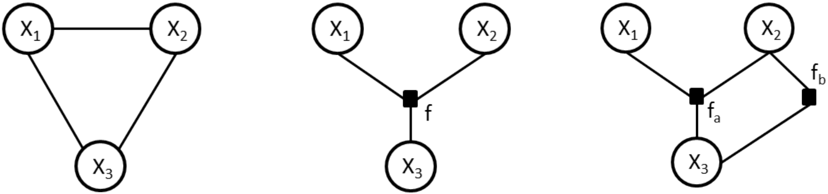
where

- f_{Ck} (or f_k) are feature functions or sufficient statistics to compute the potential of clique C
- $\theta_{Ck} \in \mathbb{R}$ are parameters
- k indexes over the available feature functions

Undirected graphical models do not express the factorization of potentials into feature functions $\Rightarrow$ factor graphs

Factor Graphs

- RV are again circular nodes
- Factors f_{Ck} are denoted as square nodes
- Edges connect a factor to the RV



$$\psi(X_1, X_2, X_3) = f(X_1, X_2, X_3)$$

$$\psi(X_1, X_2, X_3) = f_a(X_1, X_2, X_3) f_b(X_2, X_3)$$

Sum-Product Inference

- A powerful class of exact inference algorithms
- Use factor graph representation to provide a unique algorithm for directed/undirected models
- Inference is feasible for chain and tree structures
  - Forward-backward algorithm in HMMs
  - Computationally more impacting in MRF due to partition function
- Inference in general MRF
  - Restructure the graph to obtain a tree-like structure to perform message passing (junction tree algorithm)
  - Approximated inference (variational, sampling)

Constrain the MRF to obtain tractable classes of undirected models

Restricting to Conditional Probabilities

In ML a part of the random variables can be assumed to be always observable $\Rightarrow$ input data

- $\mathbf{X}_k$ - observable inputs in the factor k
- $\mathbf{Y}_k$ - hidden (or partly observable) RV
- $f_k(\mathbf{X}_k, \mathbf{Y}_k)$ - factor feature function

Under this assumption we can directly model the conditional distribution

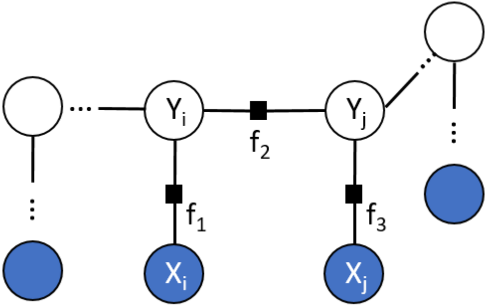
$$P(\mathbf{Y}|\mathbf{X}) = \frac{1}{Z(\mathbf{X})} \prod_k \exp\{\theta_k f_k(\mathbf{X}_k, \mathbf{Y}_k)\}$$

where $\mathbf{X}$ is the joint input that is always available

$$Z(\mathbf{X}) = \sum_{\mathbf{y}} \prod_k \exp\{\theta_k f_k(\mathbf{X}_k, \mathbf{Y}_k = \mathbf{y}_k)\}$$

Conditional Random Field (CRF)

Constrained MRF models representing input-conditional distributions

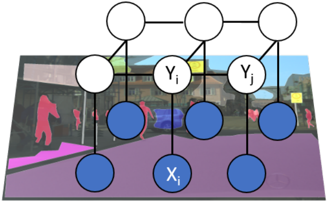


$$P(\mathbf{Y}|\mathbf{X},\theta) = \frac{1}{Z(\mathbf{X})}\exp(\theta_1 f_1(X_i,Y_i)+\theta_2 f_2(Y_i,Y_j)+\theta_3 f_(X_j,Y_j)+\dots)$$

Feature functions

What does a feature function $f_k(\mathbf{X}_k, \mathbf{Y}_k)$ do?

- Represent couplings or constraints between random variables
- Often very simple, such as linear functions



- Make noisy binary pixel X_i and its clean version Y_i have same sign

$$f_i(X_i, Y_i) = X_i Y_i$$

- Constrain nearby interpretations to be similar

$$f_{ij}(Y_i, Y_j) = Y_i^T Y_j$$

Discriminative Learning in Graphical Models

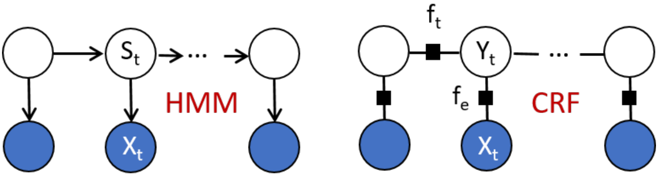
X is always observable input while **Y** can be unobserved

- Let us simplify the problem by considering to have a single Y and multiple **X**
- Let us assume that we can observe the Y^n corresponding to $\mathbf{X}^n$ for some samples n
- We can use this information to fit θ in $P(Y|\mathbf{X}, \theta)$
- What does $P(Y|\mathbf{X}', \theta)$ do for a new $\mathbf{X}'$ sample without observable Y'? Performs a prediction (e.g. classification if Y is multinomial)

The model above describes the Logistic Regression/Classifier: a discriminative version of Naive Bayes

A CRF for Sequences

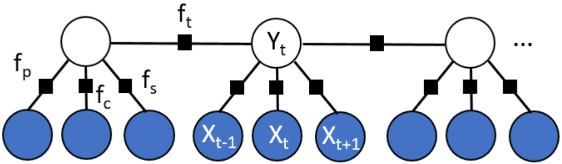
The undirected and discriminative equivalent of an HMM



Is this all about substituting emission probability with feature f_e and transition distribution with f_t?

A Generalization of HMM

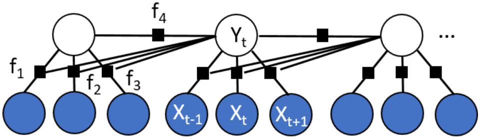
Modeling relative influence of suffix and prefix symbols



$$P(\mathbf{Y}|\mathbf{X},\theta) = \frac{1}{Z(\mathbf{X})} \prod_t \exp\{\theta_p f_p(X_{t-1}, Y_t) + \theta_e f_e(X_t, Y_t) +$$
$$\theta_s f_s(X_{t+1}, Y_t) + \theta_t f_t(Y_{t-1}, Y_t)\}$$

Generic LCRF Formulation

Modeling explicitly input influence on transition



General Linear CRF Likelihood:

$$P(\mathbf{Y}|\mathbf{X},\theta) = \frac{1}{Z(\mathbf{X})} \prod_t \exp\left\{ \sum_k \theta_k f_k(Y_t, Y_{t-1}, \mathbf{X}_t) \right\}$$

Use indicator variables in f_k definition to include or disregard the influence of specific RV, e.g. $\mathbb{1}_{Y_t=i}\mathbb{1}_{X_t=o}$

Posterior Inference in LCRF

Is there an equivalent of the smoothing problem in LCRF? Yes: $P(Y_t, Y_t - 1|\mathbf{X})$

- Solved by (exact) forward-backward inference
- Sum-product message passing on the LCRF factor graph

$$P(Y_t, Y_t - 1|\mathbf{X}) \propto \alpha_{t-1}(Y_{t-1})\psi_t(Y_t, Y_{t-1}, X_t)\beta_t(Y_t)$$

Clique weighting

$$\psi_t(Y_t, Y_{t-1}, X_t) = \exp\{\theta_e f_e(X_t, Y_t) + \theta_t f_t(Y_{t-1}, Y_t)\}$$

Forward Message

$$\alpha_t(i) = \sum_j \psi_t(i, j, X_t)\alpha_{t-1}(j)$$

Backward Message

$$\beta_t(j) = \sum_i \psi_{t+1}(i, j, X_{t+1})\beta_{t+1}(i)$$

Other Inference Problems

- Max-product inference can be performed as in the Viterbi algorithm for HMM
- The computationally expensive part is the computation of exponential summation in $Z(\mathbf{X})$ term
  - The forward-backward algorithm computes it efficiently as normalization term of $P(Y_t, Y_t - 1|\mathbf{X})$
- Exact inference in CRF other than chain-like is likely to be computationally impractical
  - Markov Chain Monte Carlo (sample y rather than estimate $P(y)$)
  - Variational Belief Propagation (reduce to message passing on trees)

Training LCRF

Maximum (conditional) log-likelihood

$$\max_{\theta} \mathcal{L}(\theta) = \max_{\theta} \sum_{n=1}^{n} \log P(\mathbf{y}^n | \mathbf{x}^n, \theta)$$

Substituting LCRF conditional formulation

$$\mathcal{L}(\theta) = \sum_{n} \sum_{t} \sum_{k} \theta_k f_k(Y_t^n, Y_{t-1}^n, \mathbf{X}_t^n) - \sum_{n} \log Z(\mathbf{X}^n) - \sum_{k} \frac{\theta_k^2}{2\sigma^2}$$

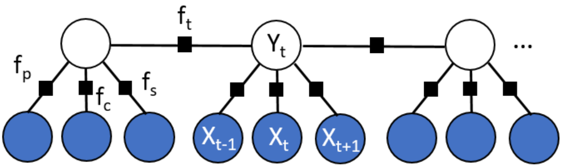
Penalized with a regularization term, e.g. based on $\|\theta\|_2$

Optimizing the Likelihood

- Typically $\mathcal{L}(\theta)$ cannot be maximized in closed form
- Use partial derivatives

$$\frac{\partial \mathcal{L}(\theta)}{\partial \theta_k} = \sum_{n,t} f_k(Y_t^n, Y_{t-1}^n, \mathbf{X}_t^n) - \sum_{n,t} \sum_{y,y'} f_k(y, y', \mathbf{X}_t^n) P(y, y' | \mathbf{X}^n) - \frac{\theta_k}{\sigma^2}$$

- First term is $\mathbb{E}[f_k]$ under the empirical distribution (i.e. with y, y' clamped)
- Second term is the $\mathbb{E}[f_k]$ under model distribution
- When gradient is zero these are equal (apart for regularization)



Stochastic Gradient Descent

In practice we can learn the θ parameters by SGD (or variants)

$$\theta^m = \theta^{m-1} - \nu_m \nabla \mathcal{L}_n(\theta^{m-1})$$

where

$$\nabla \mathcal{L}_{nk}(\theta) = \sum_t f_k(Y_t^n, Y_{t-1}^n, \mathbf{X}_t^n) - \sum_t \sum_{y,y'} f_k(y, y', \mathbf{X}_t^n) P(y, y' | \mathbf{X}^n) - \frac{\theta_k}{N\sigma^2}$$

and $P(y, y'|\mathbf{X}^n)$ is estimated by sum-product inference

Engineering Features

Linear CRF have found wide applications

- Text processing: POS-tagging, semantic role identification
- Bioinformatics: sequence alignment, protein structure prediction

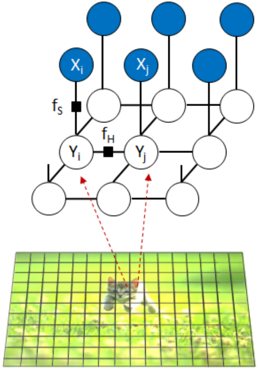
Feature functions have often the form $f_k(\mathbf{X}_k, \mathbf{Y}_k) = \mathbb{1}_{\mathbf{y}_k = \hat{\mathbf{y}}_k} q(\mathbf{X}_c)$

- f_k is non-zero only for a specific output configuration $\hat{\mathbf{y}}_k$
- f_k then depends only on $\mathbf{X}_k$ (i.e. parameters are not shared by classes)

Observation functions $q(\mathbf{X}_c)$: word begins with capital, ends with -ing, ...

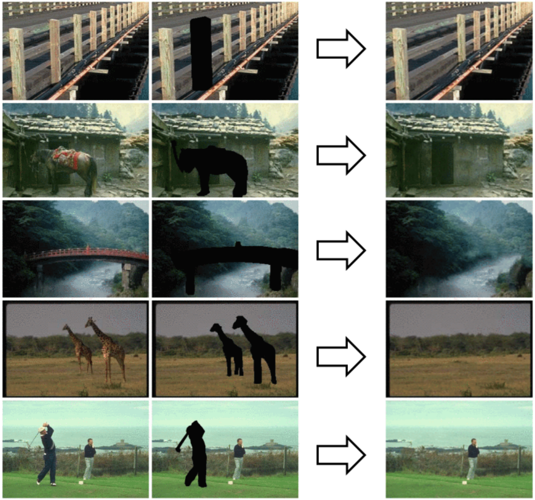
MRF/CRF in Vision

- Define bi-dimensional lattice on the image
  - Regular grid, patches, superpixels, segments
- Background/Foreground segmentation
  - X_i Observable label
  - Y_i Region annotation as background/foreground
- Impose constraints
  - $f_S(Y_i, X_i) \Rightarrow$ Cost of disregarding available annotation
  - $f_H(X_i, X_j) \approx [x_i \neq x_j]w_{ij} \Rightarrow$ Label affinity constraint weighted by region similarity w_{ij}



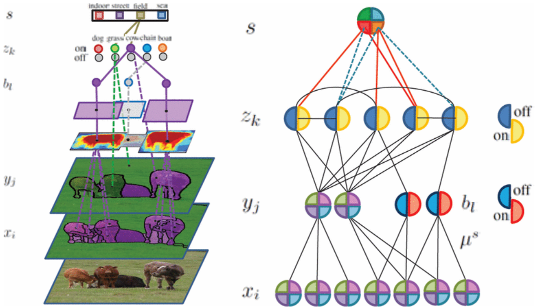
Background Segmentation

Image Completion



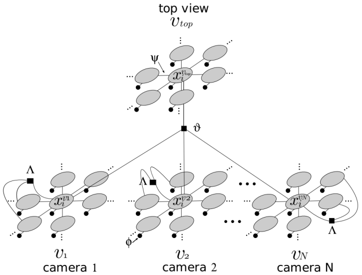
N. Komodakis. Image Completion Using Global Optimization. CVPR 2006

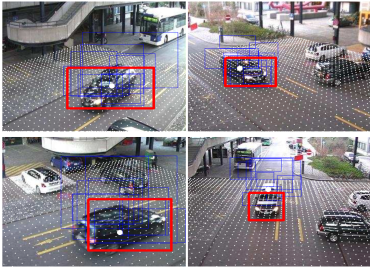
Semantic Segmentation



J. Yao, S. Fidler and R. Urtasun, "Describing the scene as a whole: Joint object detection, scene classification and semantic segmentation," ICCV 2012

Integrating Prior Information





Roig et al "Conditional Random Fields for multi-camera object detection," ICCV 2011

MRF Software

- CRFsuite - Fast implementation of linear/chain CRFs for NLP applications (native C++; Scikit-like package python-crfsuite)
- PyStruct - Python CRF package including 2D lattices, graph structures and several inference algorithms
- pgmpy - Python library for graphical models (includes CRF, MRF and more)
- Pyro - Ubers' own PyTorch provide an implementation of Deep CRF
- UGM - Matlab library for Markov Random Fields
- CRF implementations (in particular linear) are present in major DL libraries (e.g. Tensorflow, PyTorch)

A Python Example

```
from pgmpy.models import MarkovModel
from pgmpy.factors.discrete import DiscreteFactor
import numpy as np
from pgmpy.inference import BeliefPropagation
...
MM=MarkovModel();
# Add edges (and nodes if not existent)
MM.add_edges_from([('f1', 'f2'), ('f2', 'f3'),('o1','f1'),('o2','f2'),('o3','f3')])

#Generate transition feature
transition=np.array([10, 90, 90, 10]);
#Generate corresponding factor
factorH1= DiscreteFactor(['f1','f2'],cardinality=[2, 2], values=transition)
#Add it to the model
MM.add_factors(factorH1)

#Solve smoothing by belief propagation (i.e. estimate hidden RV)
belief_propagation = BeliefPropagation(MM)
ymax=belief_propagation.map_query(variables=['f1','f2','f3'],\
evidence={'o1' : toVal('class1'), 'o2' : toVal('class1'), 'o3' : toVal('class2')})
...
```

Take Home Messages

- Markov Random Fields
  - Undirected graphical models
  - Allow to express constraints between RV without needing to use probabilities
  - Topology follows data structure/relations and allow embedding prior information
- Conditional Random Fields
  - Constrained MRF learning discriminative posteriors
  - Feature functions to model constraints (often simple hand-coded feature detectors)
  - Parameters allow to linearly combine features
- CRF/MRF are often used as final refinement (segmentation, POS tagging, ...)

Next Lecture

Boltzmann Machines

- A first bridge between (undirected) generative models and (recurrent) neural networks
- Restricted Boltzmann Machines
- Contrastive Divergence training