Lezione / Esercitazione 06/05

$$A = \begin{bmatrix} \alpha & 1 & & O \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ \alpha & \cdots & & \alpha \end{bmatrix} \in \mathbb{R}^{n \times n} \qquad n \geq 2$$

p.d. ?

$k = 1 \quad |\alpha| > 1$

$k = 2 \quad |\alpha| > |\alpha| + 1 \quad (n > 2)$

$|\alpha| > |2| \quad (n = 2)$

per nessun valore.

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per quali valori di α il metodo di Gauss-Seidel è convergente?

$$A = \begin{bmatrix} \alpha & 1 & & O \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ \alpha & \cdots & & \alpha \end{bmatrix} \quad M = \begin{bmatrix} \alpha & & \\ \vdots & \ddots & \\ \alpha & \cdots & \alpha \end{bmatrix} \quad N = \begin{bmatrix} 0 & -1 & & \\ & & \ddots & \\ & & & -1 \\ & & O & \end{bmatrix}$$

slabě stabilní: $\varphi(G) = \max\limits_{i=1\ldots n} |\alpha_i|$

$$G = \begin{bmatrix} \alpha & & \\ \vdots & \ddots & \\ \alpha & \cdots & \alpha \end{bmatrix}^{-1} \begin{bmatrix} -1 & & \\ & \ddots & \\ 0 & & -1 \end{bmatrix}$$

$$\left(\alpha \begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{bmatrix} \right)^{-1} = \frac{1}{\alpha} \begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 0 & -1 & 1 & \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

Tedy:

$$\begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -1 & \ddots & \\ & \ddots & \\ & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$M^{-1} = \frac{1}{\alpha} \begin{bmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}$$

$$G = M^{-1}N = \frac{1}{\alpha} \begin{bmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & 0 \end{bmatrix}$$

$$= \frac{1}{\alpha} \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 \cdots & 0 \\ 0 & 0 & 1 & -1 & \\ & & & \ddots & \ddots \\ & & & & \ddots \end{bmatrix} = \begin{bmatrix} 0 & -1/\alpha & & \\ & \frac{1}{\alpha} & \ddots & \\ & & \ddots & -1/\alpha \\ & & & \frac{1}{\alpha} \end{bmatrix}$$

G triangolo superiore $\Rightarrow$ gli autovalori di G stanno sulla diagonale principale. $G < \begin{matrix} 0 \\ \frac{1}{\alpha} \end{matrix}$

$$\rho(G) = \frac{1}{|\alpha|}$$

$$\frac{1}{|\alpha|} < 1 \iff 1 < |\alpha| \iff |\alpha| > 1$$

$$\det(\delta I - M^{-1}N) = \det(\delta M^{-1} \cdot M - M^{-1}N)$$

$$= \det\left(M^{-1} \cdot (\delta M - N)\right) \overset{\beta_{\text{mat}}}{=}$$

$$= \det M^{-1} \det(\delta M - N)$$

$$\det(\delta I - G) = 0 \iff \det(\delta M - N) = 0$$

$$\det\left(\alpha\delta \begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & \\ & \ddots & \ddots \\ & & 1 \end{bmatrix}\right)$$

$$= \det \begin{pmatrix} \alpha\delta & 1 & & \\ \alpha\delta & \alpha\delta & \ddots & \\ \vdots & & \ddots & 1 \\ \alpha\delta & \cdots & \cdots & \alpha\delta \end{pmatrix} = 0 \; ?$$

$$\det(M^{-1}N) = \prod_{i=1}^{n} \lambda_i \qquad \lambda_i \text{ autoval } M^{-1}N$$

$$\det M^{-1} \cdot \det N = \frac{\det N}{\det M}$$

$$M = \begin{bmatrix} 2 & & \\ \vdots & \ddots & \\ 2 & \cdots & 2 \end{bmatrix}$$

$$M^{-1} = \frac{1}{2}\begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{bmatrix}$$

$$= \frac{\det N}{2^n} \qquad \lambda_1 \cdots \lambda_n = \frac{0}{2^n} = 0$$

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ & \ddots & & \\ \frac{1}{2} & \cdots & \frac{1}{2} & 1 \end{bmatrix} \qquad M = I, \quad N = \begin{bmatrix} 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ & \ddots & & \\ & & \ddots & \\ -\frac{1}{2} & \cdots & -\frac{1}{2} & 0 \end{bmatrix}$$

$$T = M^{-1}N = N = \begin{bmatrix} 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ & 0 & & \\ -\frac{1}{2} & \cdots & -\frac{1}{2} & 0 \end{bmatrix} \qquad n > 2$$

$$\|T\|_1 = 1 \qquad \|T\|_\infty = \frac{1}{2}(n-1)$$

$\varphi(J)$

$$\det(\lambda I - J) = \det\begin{pmatrix} \lambda & \frac{1}{2} & \cdots & \frac{1}{2} \\ & \ddots & & \\ \frac{1}{2} & \cdots & \frac{1}{2} & \lambda \end{pmatrix}$$

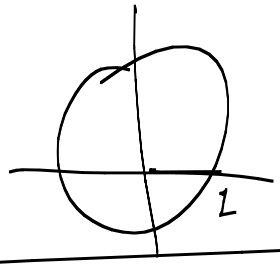
$$= \lambda \det\begin{pmatrix} \lambda & \frac{1}{2} & \cdots & \frac{1}{2} \\ & \ddots & & \\ \frac{1}{2} & \cdots & \frac{1}{2} & \lambda \end{pmatrix} = \cdots = \lambda^{n-2} \det\begin{pmatrix} \lambda & \frac{1}{2} \\ \frac{1}{2} & \lambda \end{pmatrix}$$

$$= \lambda^{n-2} \cdot \left(\lambda^2 - \frac{1}{4}\right)$$

$\lambda^{n-2} = 0 \Leftrightarrow \lambda = 0$

$\lambda^2 - \frac{1}{4} = 0 \Leftrightarrow \lambda = \pm \frac{1}{2}$

$\varphi(J) = \frac{1}{2} < 1 \Rightarrow$ Jacobi è convergente



$$J^2 = \begin{bmatrix} 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ & & & \\ -\frac{1}{2} & \cdots & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ & & & \\ -\frac{1}{2} & \cdots & -\frac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & & \frac{1}{4} & 0 \\ & & 0 & & \\ 0 & \frac{1}{4} & \cdots & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \qquad \|J^2\|_1 = \frac{1}{2}$$

$$\frac{\|e^{(2k)}\|_1}{\|e^{(0)}\|_1} \leq 2^{-k}$$

$$e^{(j)} = P^j \cdot e^{(0)} \qquad j \geq 0$$

$$e^{(2k)} = J^{2k} \cdot e^{(0)}$$

$$\|e^{(2k)}\|_1 = \|J^{2k} \cdot e^{(0)}\|_1$$

$$\leq \| J^{2k} \|_1 \| e^{(0)} \|_1$$

$$J^{2k} = \underbrace{J^2 \cdot J^2 \cdot \ldots \cdot J^2}_{k \text{ volte}}$$

$$\| J^{2k} \|_1 = \| J^2 \cdots J^2 \|_1 \leq \| J^2 \|^k$$

$$\| e^{(2k)} \|_1 \leq \| J^2 \|_1^k \cdot \| e^{(0)} \|_1$$

$$\leq 2^{-k} \| e^{(0)} \|_1$$

$$\frac{\| e^{(2k)} \|_1}{\| e^{(0)} \|_1} \leq 2^{-k}$$

$f(x) = 0 \qquad f : \mathbb{R} \to \mathbb{R}$

$f : [a, b] \to \mathbb{R}$

$f(x) = e^x - \log x \qquad f(x) = x - \cos x$

$f(x) = e^x - 3x \qquad f(x) = x^3 - \cancel{3}x + 2$

(1) Se esistono, dove e quante sono le soluzioni dell'equazione.

RISPOSTA ANALISI GRAFICA

$f(x) = e^x - 3x = 0$

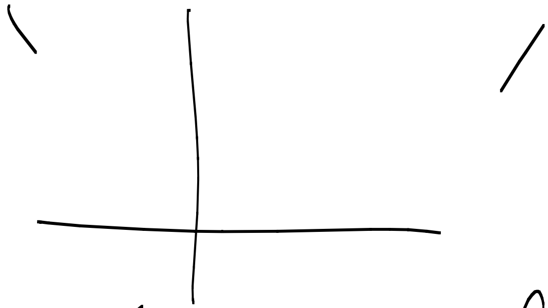
$f(x) \in C^\infty(\mathbb{R})$

$\lim_{x \to +\infty} f(x) = +\infty - \infty$ con de indeterminazione.

$$\lim_{x\to+\infty} e^x - 3x = \lim_{x\to+\infty} e^x \cdot \underbrace{\left(1 - \frac{3x}{e^x}\right)} = +\infty$$

$$\lim_{x\to+\infty} \frac{3x}{e^x} \overset{\text{Hopital}}{=} \lim_{x\to+\infty} \frac{3}{e^x} = 0$$

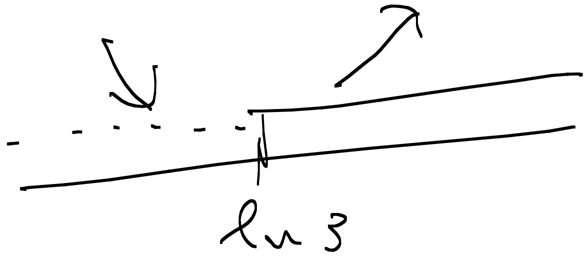
$$\lim_{x\to-\infty} e^x - 3x = +\infty$$

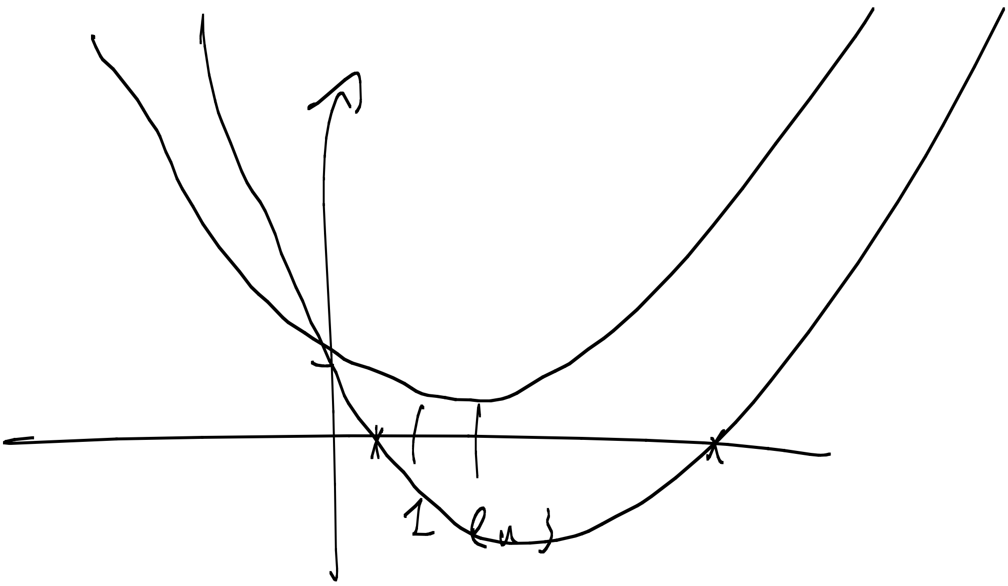


$$f'(x) = e^x - 3 \qquad f'(x) \geq 0 \iff$$

$$e^x - 3 \geq 0 \iff e^x \geq 3$$

$$x \geq \ln 3$$



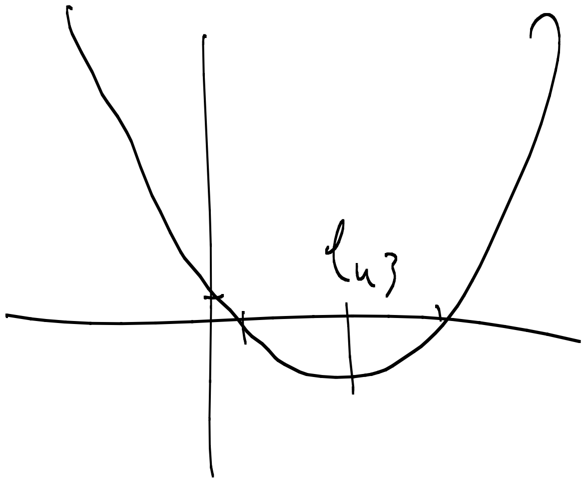


$$f(0) = e^0 - 0 = 1$$

$$f(\ln 3) = e^{\ln 3} - 3\ln 3$$

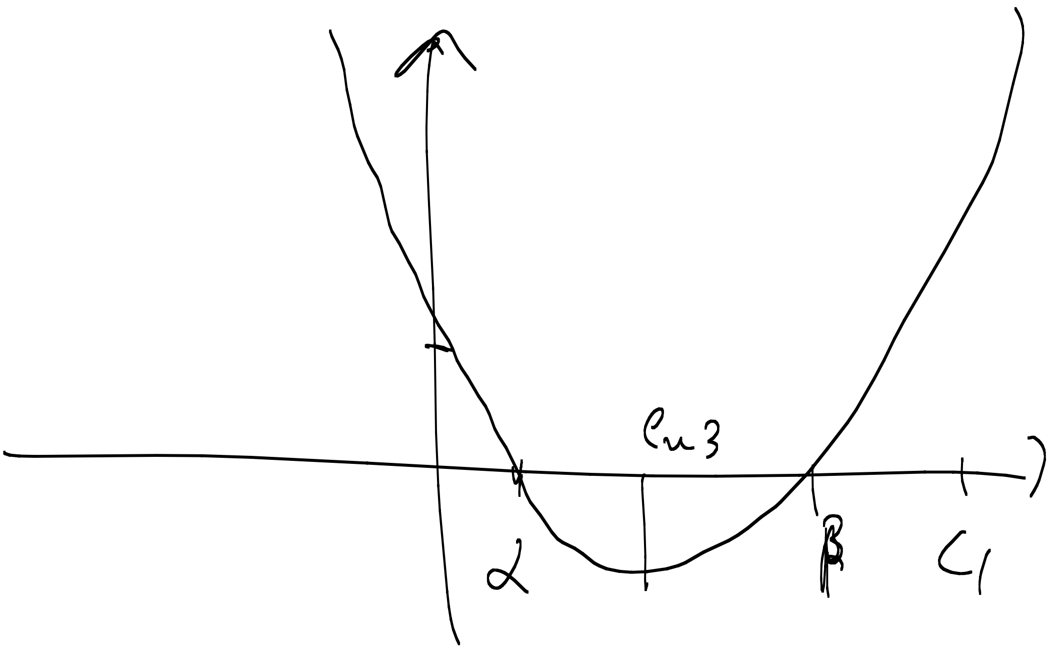
$$= 3 - 3\ln 3 = 3(1 - \ln 3)$$

$$< 0$$



2 soluzioni reali — localizzo le soluzioni $\rightarrow f(\alpha)=0$

1 determino $[a,b]$ t.c. $\alpha \in [a,b]$ e α è l'unica soluzione in $[a,b]$



$\alpha \in [0, \ln 3]$

$f(2) = e^2 - 3 \cdot 2$ $\qquad$ $f(4) = e^4 - 12 > 0$

$f(3) = e^3 - 3 \cdot 3$ $\qquad$ $\beta \in [\ln 3, 4]$

$f(x) = e^x - 3x = 0$

$e^x - 3x = 0 \iff \underline{e^x} = \underline{3x}$

$$\begin{cases} y = e^x \\ y = 3x \end{cases}$$