CORREZIONE SIMULAZIONE DI PROVA SCRITTA

$$A = \begin{bmatrix} -2 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \alpha & & & -2 \end{bmatrix}$$

- α pred. diagonale ?
- α ∃! LU → LU
- α invertibile
- α Jacobi è convergente

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \qquad i=1\dots n$$

$i = 1 : n-1 \qquad |-2| = 2 > |1| = 1 \quad$ ok

$i = n \qquad |-2| = 2 > |\alpha| \iff |\alpha| < 2 \iff -2<\alpha<2$

A p.di. (per righe) $\iff -2<\alpha<2$

$$A = \begin{bmatrix} -2 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \alpha & & & -2 \end{bmatrix} \qquad A[1:n-1,\, 1:n-1] = \begin{bmatrix} -2 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & -2 \end{bmatrix}$$

$A(1:k, 1:k)$ è invertibile per $k=1\dots n-1$

$\Rightarrow \forall \alpha \in \mathbb{R} \quad \exists!$ LU di A

$$A = \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline v^T & 1 \end{array}\right] \left[\begin{array}{cccc|c} -2 & 1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & & 1 & 0 \\ & & & -2 & 1 \\ \hline & 0^T & & & x \end{array}\right]$$

$$v^T \begin{bmatrix} -2 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & -2 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & \cdots & 0 \end{bmatrix}$$

$$\begin{bmatrix} v_1 & \cdots & v_{n-1} \end{bmatrix} \begin{bmatrix} -2 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & -2 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & \cdots & 0 \end{bmatrix}$$

$$-2v_1 = \alpha \iff v_1 = -\frac{\alpha}{2}$$

$$v_1 - 2v_2 = 0 \iff 2v_2 = v_1 \iff v_2 = -\frac{\alpha}{4}$$

$$v_2 - 2v_3 = 0 \iff 2v_3 = v_2 \iff v_3 = -\frac{\alpha}{8}$$

$$v_k = -\frac{\alpha}{2^k} \qquad k = 1 \dots n-1$$

$$v_{n-1} + x = -2$$

$$x = -2 - v_{n-1} = -2 + \frac{\alpha}{2^{n-1}}$$

$$= \frac{\alpha - 2^n}{2^{n-1}}$$

③ A è invertibile ⟺ U invertibile $U = \begin{bmatrix} -2 & & & \\ & \ddots & & \\ & & -2 & \\ & & & x \end{bmatrix}$

⟺ x è invertibile ⟺ $x \neq 0$

$$x = \frac{\alpha - 2^n}{2^{n-1}} . \qquad x \neq 0 \iff \alpha \neq 2^n$$

$$\left(\det A = (-2)^{n-1} \cdot \left(\frac{\alpha - 2^n}{2^{n-1}} \right) = (-1)^{n-1} \cdot (\alpha - 2^n) \right)$$

$$A = \begin{bmatrix} -2 & +1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \alpha & & & -2 \end{bmatrix}$$

A pd ⟹ Jacobi convergente

$$M = -2I \qquad N = \begin{bmatrix} & -1 & & \\ & & \ddots & \\ & & & -1 \\ -\alpha & & & \end{bmatrix}$$

$$J = M^{-1} N = -\frac{1}{2} I N = \begin{bmatrix} & \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{2} \\ \frac{1}{2}\alpha & & & \end{bmatrix}$$

$\underline{\rho(J) = ?}$

$$\det \begin{pmatrix} \lambda & -\frac{1}{2} & & \\ & \ddots & \ddots & \\ & & \ddots & -\frac{1}{2} \\ -\frac{1}{2}\alpha & & & \lambda \end{pmatrix}$$

regola di Laplace

fattorizzazione LU simbolica.

$$= \lambda \cdot \det \begin{pmatrix} \lambda & -\frac{1}{2} & & \\ & \ddots & \ddots & \\ & & \ddots & -\frac{1}{2} \\ & & & \lambda \end{pmatrix} + (-1)^{n+1} \left(-\frac{1}{2}\alpha \right) \cdot \det \begin{pmatrix} -\frac{1}{2} & & \\ \lambda & \ddots & \\ & \ddots & -\frac{1}{2} \\ & \lambda & \end{pmatrix}$$

$$= \lambda \cdot \lambda^{n-1} + (-1)^{n+1} \cdot \left(-\frac{1}{2}\alpha\right)\left(-\frac{1}{2}\right)^{n-1}$$

$$= \lambda^{n} + (-1)^{n+1} \cdot \left(-\frac{1}{2}\right)^{n} \cdot \alpha = \qquad (ab)^n = a^n b^m$$

$$= \lambda^{n} + (-1)^{n+1} \cdot (-1)^{n} \cdot \left(\frac{1}{2}\right)^{n} \alpha$$

$$= \lambda^{n} + (-1)^{2n+1} \left(\frac{1}{2}\right)^{n} \alpha = \lambda^{n} - \left(\frac{1}{2}\right)^{n} \alpha$$

$$\lambda^{n} - \left(\frac{1}{2}\right)^{n} \alpha = 0 \Rightarrow |\lambda^{n}| = \left(\frac{1}{2}\right)^{m} \cdot |\alpha|$$

$$\Rightarrow |\lambda|^{n} = \left(\frac{1}{2}\right)^{m} |\alpha| \Rightarrow |\lambda| = \sqrt[n]{\left(\frac{1}{2}\right)^{m} |\alpha|}$$

$$\Rightarrow |\lambda| = \frac{1}{2}\sqrt[n]{|\alpha|}$$

$$|\lambda| < 1 \Leftrightarrow \frac{1}{2}\sqrt[n]{|\alpha|} < 1 \Leftrightarrow \sqrt[n]{|\alpha|} < 2 \Leftrightarrow |\alpha| < 2^{n}$$

$$-2^{n} < \alpha < 2^{n} \quad \blacksquare$$

$$\lambda^{n} + (-1)^{n+1} \cdot \left(-\frac{1}{2}\right)^{n} \alpha = 0$$

$$\Leftrightarrow \quad \epsilon^n = -(-1)^{n+1}\cdot\left(-\tfrac{1}{2}\right)^n \cdot \alpha$$

$$\Rightarrow \quad |\epsilon^n| = \left(\tfrac{1}{2}\right)^n |\alpha| \qquad \cancel{\epsilon = \sqrt[n]{-(-1)^{n+1}\left(-\tfrac{1}{2}\right)^n\alpha}}$$

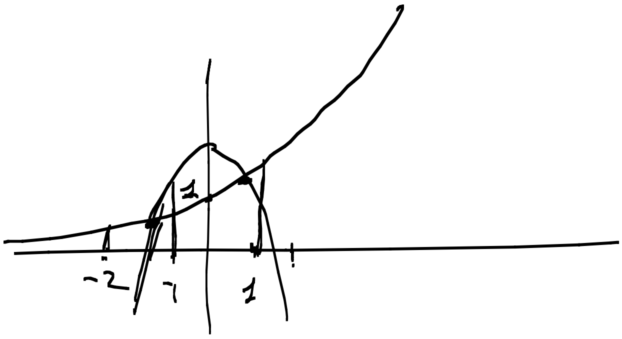
$$f(x) = e^x + x^2 - 2 = 0$$

(1) Dire quante sono e dove sono le soluzioni dell'equazione.

$$f(x) = 0 \Leftrightarrow e^x + x^2 - 2 = 0 \Leftrightarrow$$

$$e^x = 2 - x^2 \Leftrightarrow \begin{cases} y = e^x \\ y = 2 - x^2 \end{cases}$$

$$\left(\begin{aligned} &\Leftrightarrow e^x - 2 = -x^2 \\ &\Leftrightarrow e^x + x^2 = 2 \end{aligned} \right)$$



$$y = 2 - x^2$$

$$x_V = -\frac{b}{2a} = 0$$

$$y_V = 2$$

$-2x = 0 \iff x = 0$

$-2 < \alpha < -1 \qquad \alpha \in [-2, -1]$

$0 < \beta < 1 \qquad \beta \in [0, 1]$

$f(x) = e^x + x^2 - 2$

$f \in C^\infty(\mathbb{R})$

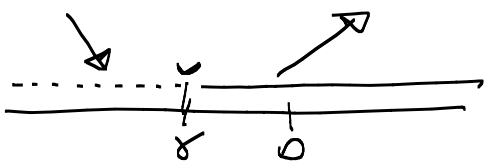
$\lim_{x \to +\infty} e^x + x^2 - 2 = +\infty$

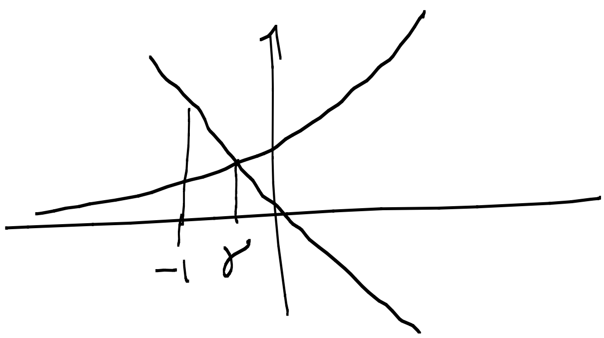
$\lim_{x \to -\infty} e^x + x^2 - 2 = +\infty$

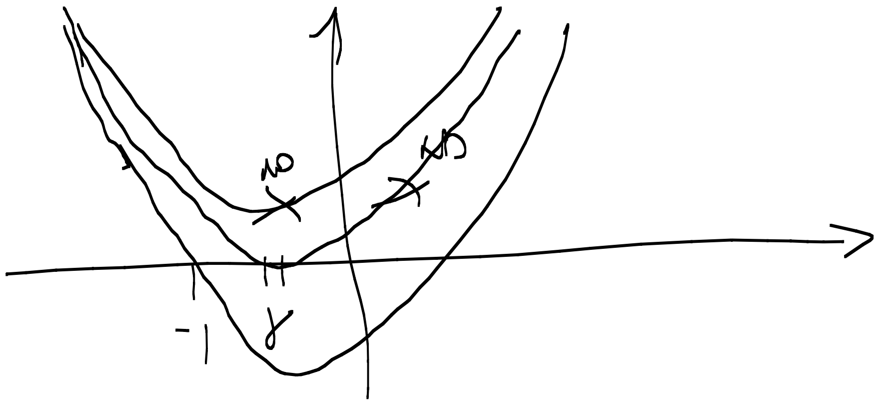
$f'(x) = e^x + 2x$

$f'(x) \geq 0 \iff e^x + 2x \geq 0 \iff$

$e^x \geq -2x$







$f(\gamma) \geq 0$? $f(\gamma) = e^{\gamma} + \gamma^2 - 2$

$f(0) = 1 + 0 - 2 = -1 < 0$

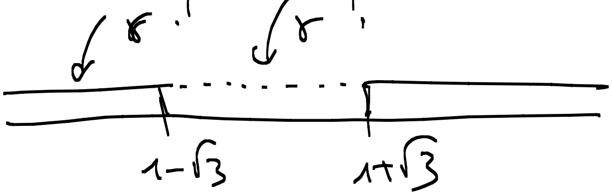
$f(\gamma) < 0$

$f(\gamma) = e^{\gamma} + \gamma^2 - 2$ $e^{\gamma} = -2\gamma$

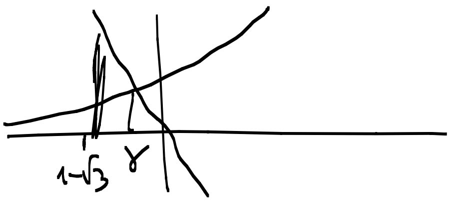
$f(\gamma) = -2\gamma + \gamma^2 - 2$

$p(x) = x^2 - 2x - 2 \geq 0$ $x = \dfrac{2 \pm \sqrt{4+8}}{2}$

$x = \dfrac{2 \pm \sqrt{12}}{2} = \dfrac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$



$1 - \sqrt{3} \simeq \underline{-0.7}$



$$\underline{-2(1-\sqrt{3})} > e^{1-\sqrt{3}}$$

$$\gamma \in [-1, 0] \overset{?}{\Longrightarrow} f(\gamma) < 0$$

$$f(\gamma) = e^{\gamma} + \gamma^2 - 2 \qquad \frac{e^{\gamma} < 1 \quad (\gamma < 0)}{\gamma^2 < 1 \quad (-1 < \gamma < 0)}$$

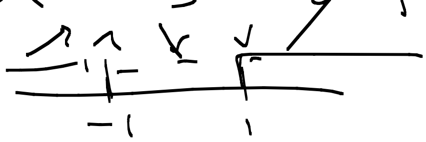
$$\Rightarrow f(\gamma) < 0$$

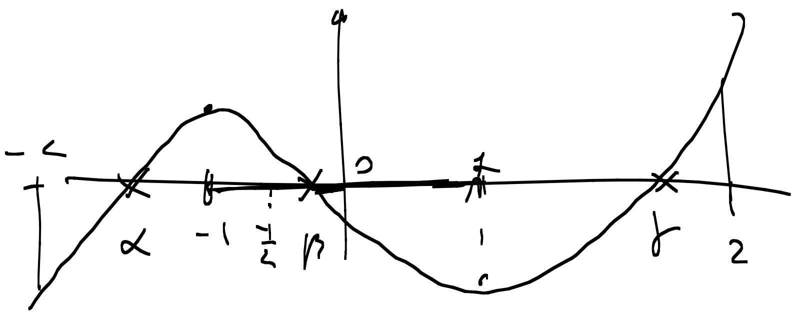
$$f(x) = x^3 - 3x - 1$$

$$f \in C^{\infty}(\mathbb{R})$$

$$\lim_{x \to +\infty} f(x) = +\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

$$f'(x) = 3x^2 - 3 \qquad f'(x) \geq 0 \Rightarrow 3 \cdot (x^2 - 1) \geq 0$$





$f(x) = x^3 - 3x - 1$

$f(-1) = -1 + 3 - 1 = 1$

$f(1) = 1 - 3 - 1 = -2$

$f(-2) = -8 + 6 - 1$

$f(2) = 8 - 6 - 1$

$\alpha \in [-2, -1)$ $\quad\quad$ $\gamma \in [1, 2]$

$\beta \in [-1, 0]$

$$x^3 - 3x - 1 = 0 \iff x^3 - 1 = 3x \iff \frac{x^3 - 1}{3} = x$$

$$\begin{cases} x_0 \in \mathbb{R} \\ x_{k+1} = g(x_k) \end{cases} \qquad g(x) = \frac{x^3 - 1}{3}$$

Th. punti fissi:
- Convergenza locale $\rightsquigarrow$ $|g'(\alpha)| < 1$
- Convergenza globale $\rightsquigarrow$ Teorema del punto fisso $|g'(x)| < 1$

$$|g'(x)| = \left| \frac{3x^2}{3} \right| = x^2$$

$$|g'(x)| < 1 \iff x^2 < 1 \iff -1 < x < 1$$

$|g'(\beta)| < 1 \Rightarrow$ in β convergenza locale.

$|g'(\alpha)| > 1 \qquad |g'(\gamma)| > 1$ $\left(\begin{array}{c}\alpha \text{ e } \gamma \\ \text{punti repulsivi}\end{array}\right)$

$[\beta - \rho \quad \beta + \rho]$ $\qquad \beta + \rho = 0 \iff \rho = -\beta$

$[2\beta, 0]$ più piccolo intervallo centrato in β che contiene 0

$2\beta > -1 \iff \beta > -\frac{1}{2}$

$f(-\frac{1}{2}) = (-\frac{1}{2})^3 + \frac{3}{2} - 1 = -\frac{1}{8} + \frac{1}{2} > 0$

$x_0 = 0$ OK /