ESERCITAZIONE 20/05

$$f(x) = \log\left(e^{x}+1\right) + x - 1 = 0$$

1. Determinare il numero di soluzioni dell'equazione e localizzarle.

2. Studiare la convergenza $x_{k+1} = 1 - \log\left(e^{x_k}+1\right) \quad k \geq 0$

   $x_{k+1} = g(x_k) \qquad g(x) = 1 - \log\left(e^{x}+1\right)$.

3. Studiare la convergenza del metodo delle tangenti.

$$f(x) = \log\left(e^{x}+1\right) + x - 1$$

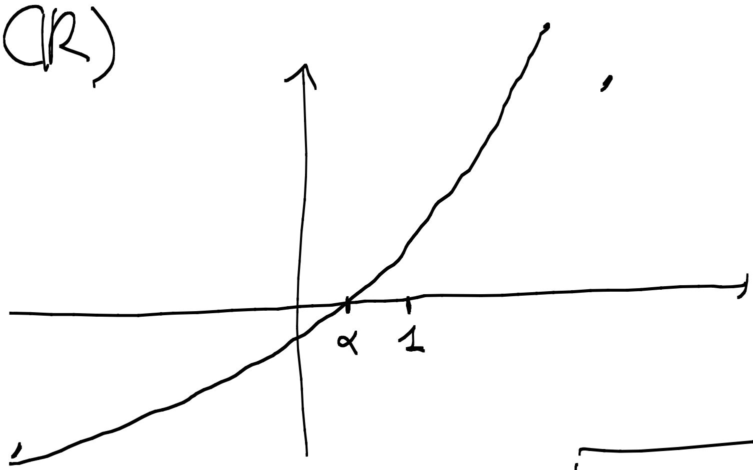
$$e^{x}+1 > 0 \quad \forall x \implies f : \mathbb{R} \to \mathbb{R}$$

$$\lim_{x \to +\infty} f(x) = +\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

$$f'(x) = \frac{1}{e^{x}+1}\, e^{x} + 1 = \frac{e^{x}}{e^{x}+1} + 1 > 0 \quad \forall x \in \mathbb{R}$$

$$f''(x) = \frac{e^x\cdot(e^x+1) - e^x\cdot e^x}{(e^x+1)^2} = \frac{e^x}{(e^x+1)^2} > 0 \ \forall_{x\in\mathbb{R}}$$

$f \in C^2(\mathbb{R})$



$f(0) = \log 2 - 1 < 0$ $\qquad \boxed{\alpha \in [0 \ 1]}$

$f(x) = \log(e^x+1) + x - 1$

$f(x) = \log(e^x+1) + x - 1 = 0$

$\log(e^x+1) + x - 1 = 0 \iff x = 1 - \log(e^x+1)$

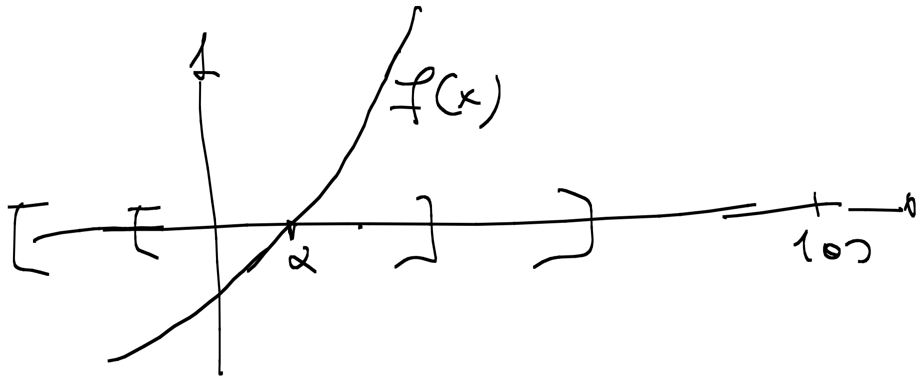
$x_{k+1} = 1 - \log(e^{x_k}+1) = g(x_k) \quad k \geq 0$

$g \in C^1(\mathbb{R})$ $\qquad g(x) = 1 - \log(e^x+1)$

$$g'(x) = -\frac{e^x}{e^x+1} \qquad |g'(x)| = \frac{e^x}{e^x+1}$$

$$\frac{e^x}{e^x+1} = |g'(x)| \overset{?}{<} 1 \quad \forall x \in \mathbb{R}$$

$$\frac{e^x}{e^x+1} < 1 \iff e^x < e^x+1$$
$$\iff 0 < 1 \quad \text{Ok } \forall x$$



$$\forall x_0 \in \mathbb{R} \qquad x_k \to \alpha$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = g(x_k)$$

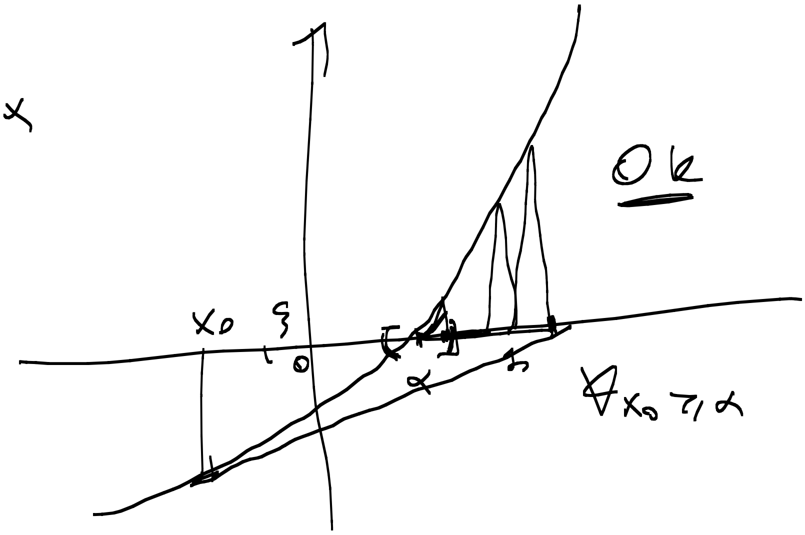
$$g(x) = x - \frac{f(x)}{f'(x)}$$

$f'(x) \neq 0 \quad \forall x$

$\Rightarrow f'(\alpha) \neq 0$

$\Rightarrow$

Convergenza locale.



Ok

$f'(x) \neq 0$

$\underline{f(x) f''(x) \geq 0}$

$\forall x_0 \geq \alpha \quad x_k \to \alpha$

$$x_1 - \alpha = g(x_0) - g(\alpha) = g'(\xi)\cdot(x_0 - \alpha)$$

$$= \frac{f(\xi) f''(\xi)}{(f'(\xi))^2}(x_0 - \alpha)$$

$x_1 - \alpha \geq 0 \quad x_1 \geq \alpha$

$\forall x_0 \in \mathbb{R} \quad x_k \to \alpha$

$$f(x) = \sqrt{x^3} - 3x + 1 = 0$$

$x^3 \geq 0 \iff \underline{x \geq 0}$

$\underline{x \geq 0}$

$$\sqrt{x^3} - 3x + 1 = 0 \iff \sqrt{x^3} = 3x - 1$$

$3x - 1 \geq 0 \qquad 3x \geq 1 \qquad \boxed{x \geq \frac{1}{3}}$

$x^3 = (3x-1)^2$ e $x \geq \frac{1}{3}$ $\sqrt{x} = -2$ $x = 4$

$f(x) = \sqrt{x^3} - 3x + 1 = x^{\frac{3}{2}} - 3x + 1$

$f: E \quad x \geq 0$

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x^{\frac{3}{2}} \cdot \left(1 - 3\underbrace{x^{-\frac{1}{2}}}_{0} + \underbrace{x^{-\frac{3}{2}}}_{0}\right) = +\infty$

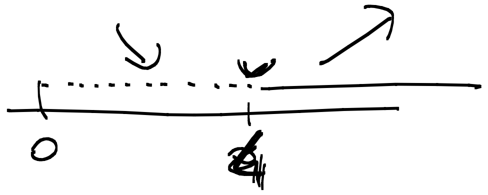
$= +\infty$

$\lim_{x \to 0^+} f(x) = f(0) = 1$

$f'(x) = \frac{3}{2} x^{\frac{3}{2}-1} - 3 = \frac{3}{2}\sqrt{x} - 3$

$= \frac{3}{2}\left(\sqrt{x} - 2\right)$

$f'(x) \geq 0 \iff \sqrt{x} - 2 \geq 0 \iff \sqrt{x} \geq 2 \iff x \geq 4$



$$f''(x) = \frac{3}{2}\cdot\frac{1}{2}\, x^{\frac{1}{2}-1} = \frac{3}{4}\, x^{-\frac{1}{2}} = \frac{3}{4}\frac{1}{\sqrt{x}} > 0$$

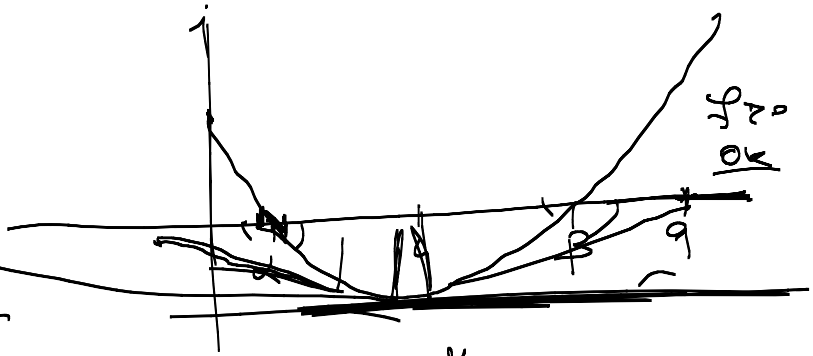
$\forall x \in \mathbb{R}^+$

$f \in C^2(\mathbb{R}^+)$ definita $x \geq 0$

C^2 per $x > 0$

$\alpha \in [0, 4]$

$\beta \in [4, 9]$ ok



$$f(4) = \sqrt{2^6} - 12 + 1 = 8 - 12 + 1 = -3$$

Convergenza locale ok $f'(\alpha) \neq 0$ $f'(\beta) \neq 0$

$x_0 \geq \beta$ $x_k \to \beta$ / $4 < x_0 \leq \beta$ $x_1 \geq \beta$

$x_k \to \beta$

$0 < x_0 \leq \alpha$ $x_k \to \alpha$

$\boxed{\alpha \leq x_0 < 4 \quad \text{ok se } \underline{x_1 > 0}}$

$x_1 = 0$?

$$x_0 - \frac{f(x_0)}{f'(x_0)} = 0$$