Markov Random Fields

INTELLIGENT SYSTEMS FOR PATTERN RECOGNITION (ISPR)

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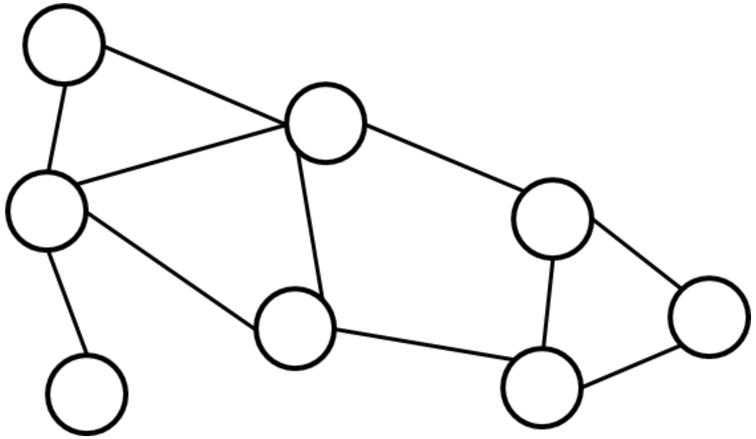
Markov Random Fields (MFRs)

- Undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ (a.k.a. Markov Networks)
- Nodes $v \in \mathcal{V}$ represent random variables X_v
  - Shaded $\Rightarrow$ observed
  - Empty $\Rightarrow$ un-observed
- Edges $e \in \mathcal{E}$ describe bi-directional dependencies between variables (constraints)



Graph often coherent with data structure

Likelihood Factorization

Define $\boldsymbol{X} = X_1, \ldots, X_N$ as the RVs associated to the N nodes in the undirected graph $\mathcal{G}$

$$P(\boldsymbol{X}) = \frac{1}{Z} \prod_{C} \psi_C(\boldsymbol{X}_C)$$

- $\boldsymbol{X}_C$ → RV associated with nodes in the maximal clique C
- $\psi_C(\boldsymbol{X}_C)$ → potential function for clique C
- Z → partition function ensuring normalization

$$Z = \sum_{\boldsymbol{X}} \prod_{C} \psi_C(\boldsymbol{X}_C)$$

Potential Functions

- Potential functions $\psi_C(\boldsymbol{X}_C)$ are not probabilities!
- Express which configurations of the local variables are preferred
- If we restrict to strictly positive potential functions, the Hammersley-Clifford theorem provides guarantees on the distribution that can be represented by the clique factorization

Definition (Boltzmann distribution)

A convenient and widely used strictly positive representation of the potential functions is

$$\psi_C(\boldsymbol{X}_C) = \exp\{-E(\boldsymbol{X}_C)\}$$

where $E(\boldsymbol{X}_C)$ is the energy function

Factorizing Potential Functions

In general, we will assume to work with MRF where the partition functions factorize as

$$\psi_C(\boldsymbol{X}_C) = \exp\left(\sum_k \theta_{Ck} f_{Ck}(\boldsymbol{X}_C)\right)$$

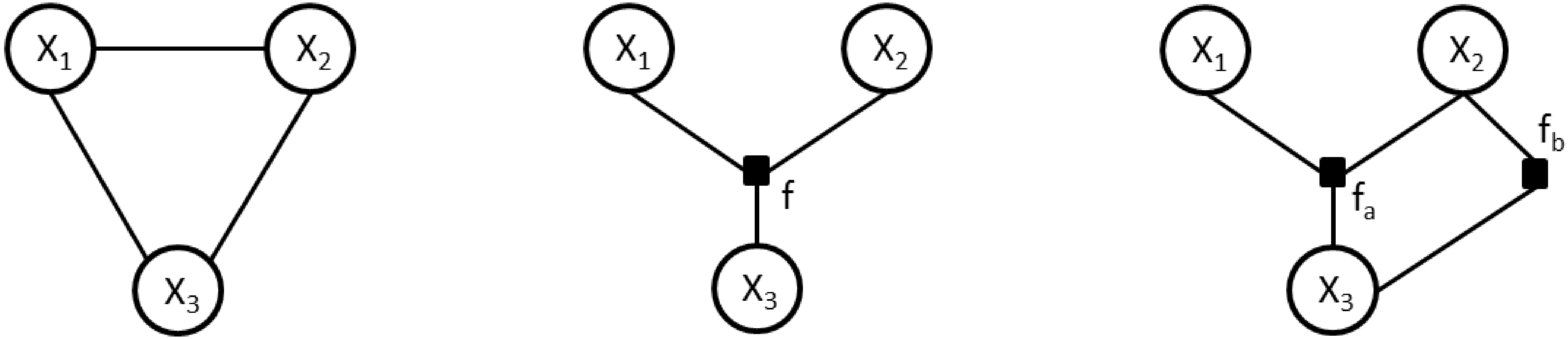
where

- f_{Ck} (or f_k) are feature functions or sufficient statistics to compute the potential of clique C
- $\theta_{Ck} \in \mathbb{R}$ are parameters
- k indexes over the available feature functions

Undirected graphical models do not express the factorization of potentials into feature functions ⇒ factor graphs

Factor Graphs

- RV are again circular nodes
- Factors f_{Ck} are denoted as square nodes
- Edges connect a factor to the RV



$$\psi(X_1, X_2, X_3) = f(X_1, X_2, X_3)$$
$$\psi(X_1, X_2, X_3) = f_a(X_1, X_2, X_3) f_b(X_2, X_3)$$

Sum-Product Inference

- A powerful class of exact inference algorithms (Belief Propagation)
- Use factor graph representation to provide a unique algorithm for directed/undirected models
- Inference is feasible for chain and tree structures
  - Forward-backward algorithm in HMMs
  - Computationally more impacting in MRF due to partition function
- Inference in general MRF
  - Restructure the graph to obtain a tree-like structure to perform message passing (junction tree algorithm, Chow-Liu)
  - Approximated inference (variational, sampling)

Constrain the MRF to obtain tractable classes of undirected models

Restricting to Conditional Probabilities

In ML a part of the random variables can be assumed to be always observable $\Rightarrow$ input data

- $\boldsymbol{X}_k$ - observable inputs in the factor k
- $\boldsymbol{Y}_k$ - hidden (or partly observable) RV
- $f_k(\boldsymbol{X}_k, \boldsymbol{Y}_k)$ - factor feature function

Under this assumption we can directly model the conditional distribution

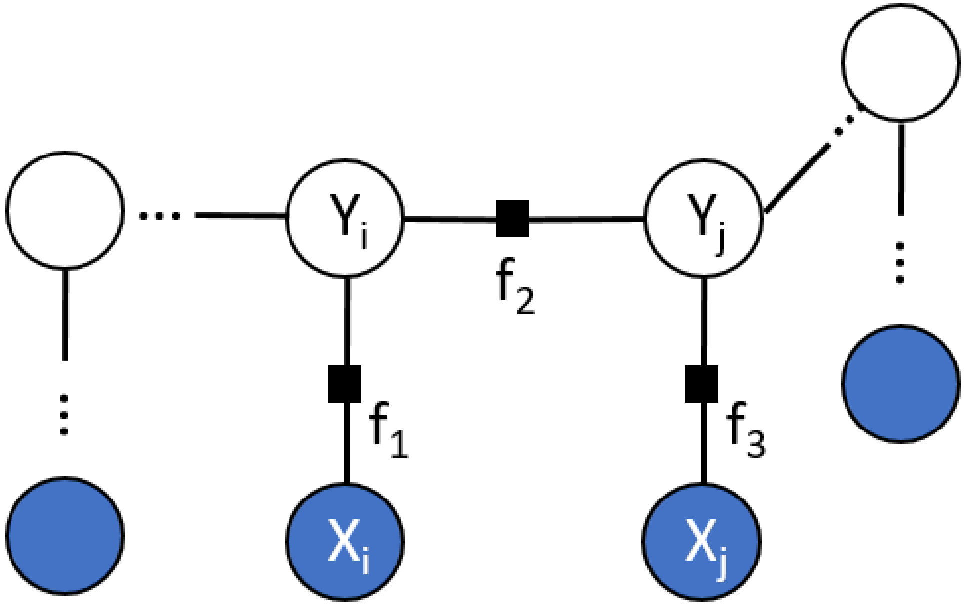
$$P(\boldsymbol{Y}|\boldsymbol{X}) = \frac{1}{Z(\boldsymbol{X})} \prod_k \exp\{\theta_k f_k(\boldsymbol{X}_k, \boldsymbol{Y}_k)\}$$

where $\boldsymbol{X}$ is the joint input that is always available

$$Z(\boldsymbol{X}) = \sum_{\boldsymbol{y}} \prod_k \exp\{\theta_k f_k(\boldsymbol{X}_k, \boldsymbol{Y}_k = \boldsymbol{y}_k)\}$$

Conditional Random Field (CRF)

Constrained MRF models representing input-conditional distributions

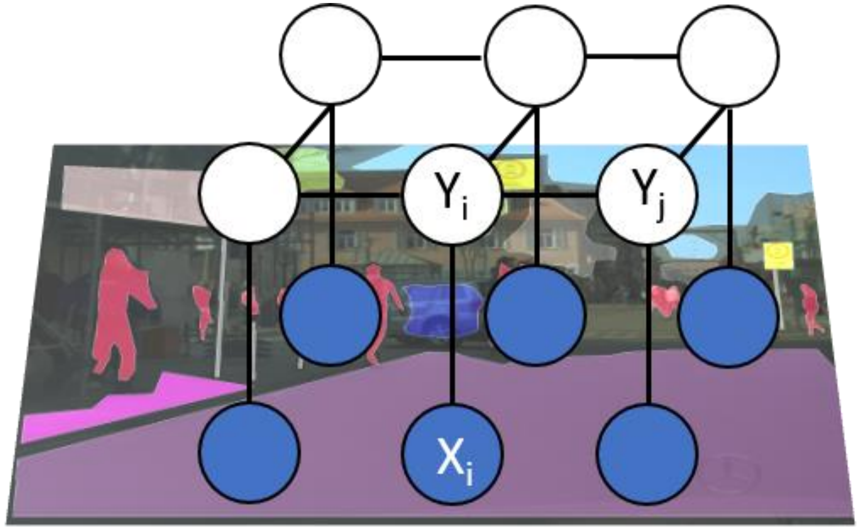


$$P(\boldsymbol{Y}|\boldsymbol{X},\theta) \ = \frac{1}{Z(\boldsymbol{X})}\exp(\theta_1 f_1(X_i\,,Y_i) + \theta_2 f_2(Y_i\,,Y_j) + \theta_3 f\,(X_j\,,Y_j)+\ldots)$$

Feature functions

What does a feature function $f_k(\boldsymbol{X}_k, \boldsymbol{Y}_k)$ do?

- Represent couplings or constraints between random variables
- Often very simple, such as linear functions



- Make noisy binary pixel Xi and its clean version Yi have same sign

$$f_i(X_i, Y_i) = X_i Y_i$$

- Constrain nearby interpretations to be similar

$$f_{ij}(Y_i, Y_j) = Y_i^T Y_j$$

Discriminative Learning in Graphical Models

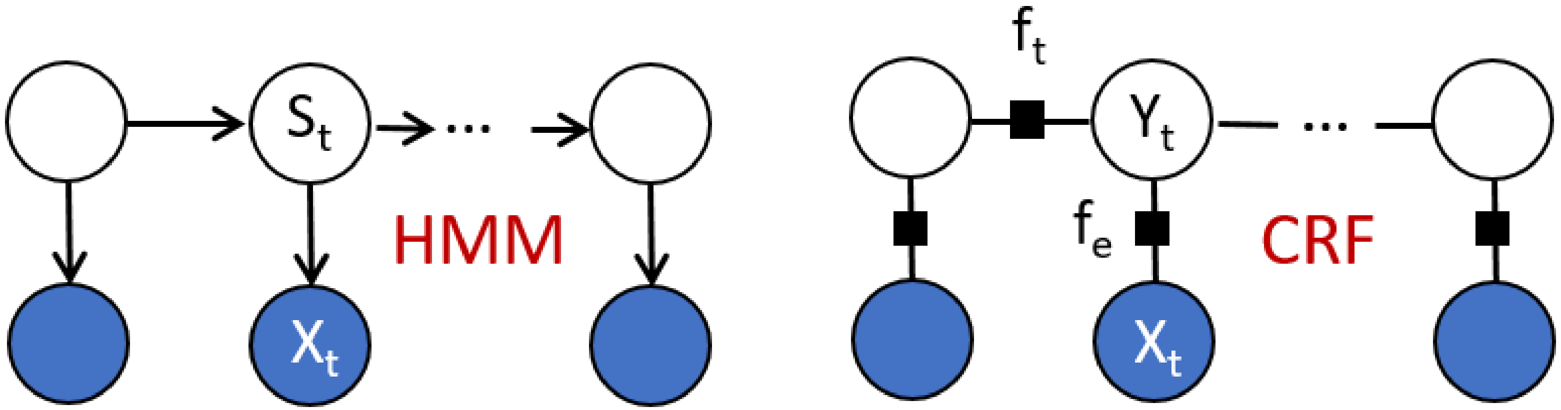
$\boldsymbol{X}$ is always observable input while $\boldsymbol{Y}$ can be unobserved

- Let us simplify the problem by considering to have a single Y and multiple $\boldsymbol{X}$
- Let us assume that we can observe the Y^n corresponding to $\boldsymbol{X}^n$ for some samples n
- We can use this information to fit θ in $P(Y|\boldsymbol{X},\theta)$
- What does $P(Y|\boldsymbol{X}',\theta)$ do for a new $\boldsymbol{X}'$ sample without observable Y'? Performs a prediction (e.g. classification if Y is multinomial)

The model above describes the Logistic Regression/Classifier: a discriminative version of Naive Bayes

A CRF for Sequences

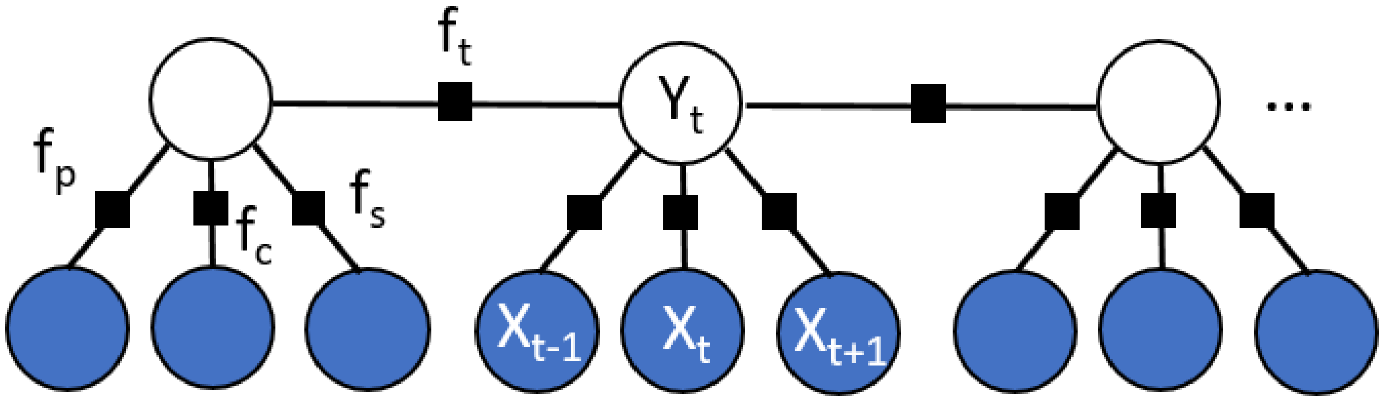
The undirected and discriminative equivalent of an HMM



Is this all about substituting emission probability with feature f_e and transition distribution with f_t?

A Generalization of HMM

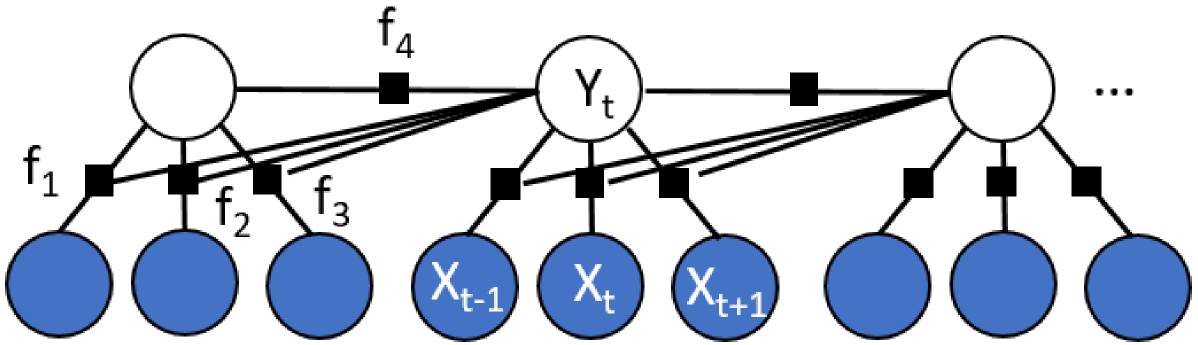
Modeling relative influence of suffix and prefix symbols



$$P(\boldsymbol{Y}|\boldsymbol{X},\theta) = \frac{1}{Z(\boldsymbol{X})}\prod_t \exp\{\theta_p f_p(X_{t-1},Y_t) + \theta_c f_c(X_t,Y_t) + \theta_s f_s(X_{t+1},Y_t) + \theta_t f_t(Y_{t-1},Y_t)\}$$

Generic LCRF Formulation

Modeling explicitly input influence on transition



General Linear CRF Likelihood:

$$P(\boldsymbol{Y}|\boldsymbol{X},\theta) = \frac{1}{Z(\boldsymbol{X})}\prod_t \prod_k \exp(\theta_k f_k(Y_t\,, Y_{t-1}, \boldsymbol{X}_t))$$

Use indicator variables in f_k definition to include or disregard the influence of specific RV, e.g. $\mathbb{1}_{Y_t=i}\ \mathbb{1}_{X_t=o}$

Posterior Inference in LCRF

Is there an equivalent of the smoothing problem in LCRF? Yes: $P(Y_t, Y_{t-1}|\boldsymbol{X})$

- Solved by (exact) forward-backward inference
- Sum-product message passing on the LCRF factor graph

$$P(Y_t, Y_{t-1}|\boldsymbol{X}) \propto \alpha_{t-1}(Y_{t-1})\psi_t(Y_t, Y_{t-1}, X_t)\beta_t(Y_t)$$

Clique weighting

$$\psi_t(Y_t, Y_{t-1}, X_t) = \exp\{\theta_e f_e(X_t, Y_t) + \theta_t f_t(Y_{t-1}, Y_t)\}$$

Forward Message

$$\alpha_t(i) = \sum_j \psi_t(i, j, X_t)\alpha_{t-1}(j)$$

Backward Message

$$\beta_t(j) = \sum_i \psi_{t+1}(i, j, X_{t+1})\beta_{t+1}(i)$$

Other Inference Problems

- Max-product inference can be performed as in the Viterbi algorithm for HMM
- The computationally expensive part is the computation of exponential summation in $Z(\boldsymbol{X})$ term
  - The forward-backward algorithm computes it efficiently as normalization term of $P(Y_t, Y_{t-1}|\boldsymbol{X})$
- Exact inference in CRF other than chain-like is likely to be computationally impractical
  - Markov Chain Monte Carlo (sample y rather than estimate $P(y)$)
  - Variational Belief Propagation (reduce to message passing on trees)

Training LCRF

Maximum (conditional) log-likelihood

$$\max_{\theta} \mathcal{L}(\theta) = \max_{\theta} \sum_{n=1}^{n} \log P(\boldsymbol{y}^n | \boldsymbol{x}^n, \theta)$$

Substituting LCRF conditional formulation

$$\mathcal{L}(\theta) = \sum_{n} \sum_{t} \sum_{k} \theta_k f_k(Y_t^n, Y_{t-1}^n, \boldsymbol{X}_t^n) - \sum_{n} \log Z(\boldsymbol{X}^n)$$

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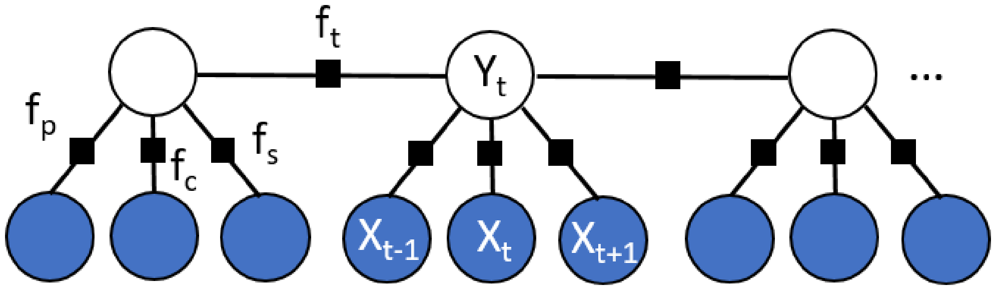
Penalized with a regularization term, e.g. based on $\|\theta\|^2$

Optimizing the Likelihood

- Typically $\mathcal{L}(\theta)$ cannot be maximized in closed form
- Use partial derivatives

$$\frac{\partial \mathcal{L}(\theta)}{\partial \theta_k} = \sum_{n,t} f_k(Y_t^n, Y_{t-1}^n, \boldsymbol{X}_t^n) - \sum_{n,\ t} \sum_{y,y\prime} f_k(y, y', \boldsymbol{X}_t^n) P(y, y' | \boldsymbol{X}^n) - \frac{\theta_k}{\sigma^2}$$

- First term is $\mathbb{E}[f_k]$ under the empirical distribution (i.e. with y, y' clamped)
- Second term is the $\mathbb{E}[f_k]$ under model distribution
- When gradient is zero these are equal (apart for regularization)



Stochastic Gradient Descent

In practice we can learn the θ parameters by SGD (or variants)

$$\theta^m = \theta^{m-1} - \nu_m \nabla \mathcal{L}_n(\theta^{m-1})$$

where

$$\nabla \mathcal{L}_{nk}(\theta) = \sum_t f_k(Y_t^n, Y_{t-1}^n, \boldsymbol{X}_t^n) - \sum_t \sum_{y,y\prime} f_k(y, y', \boldsymbol{X}_t^n) P(y, y' | \boldsymbol{X}^n) - \frac{\theta_k}{N\sigma^2}$$

and $P(y, y' | \boldsymbol{X}^n)$ is estimated by sum-product inference

Engineering Features

Linear CRF have found wide applications

- Text processing: POS-tagging, semantic role identification
- Bioinformatics: sequence alignment, protein structure prediction

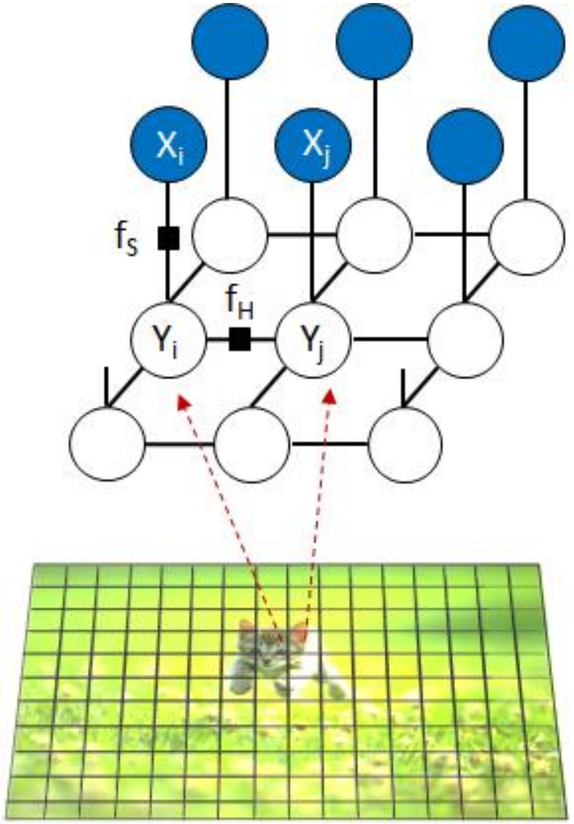
Feature functions have often the form $f_k(\boldsymbol{X}_k, \boldsymbol{Y}_k) = \mathbb{1}_{\boldsymbol{y}_k = \widehat{\boldsymbol{y}}_k} q(\boldsymbol{X}_c)$

- f_k is non-zero only for a specific output configuration $\widehat{\boldsymbol{y}}_k$
- f_k then depends only on $\boldsymbol{X}_k$ (i.e. features are not shared by classes)

Observation functions $q(\boldsymbol{X}_c)$: word begins with capital, ends with -ing, ...

MRF/CRF in Vision

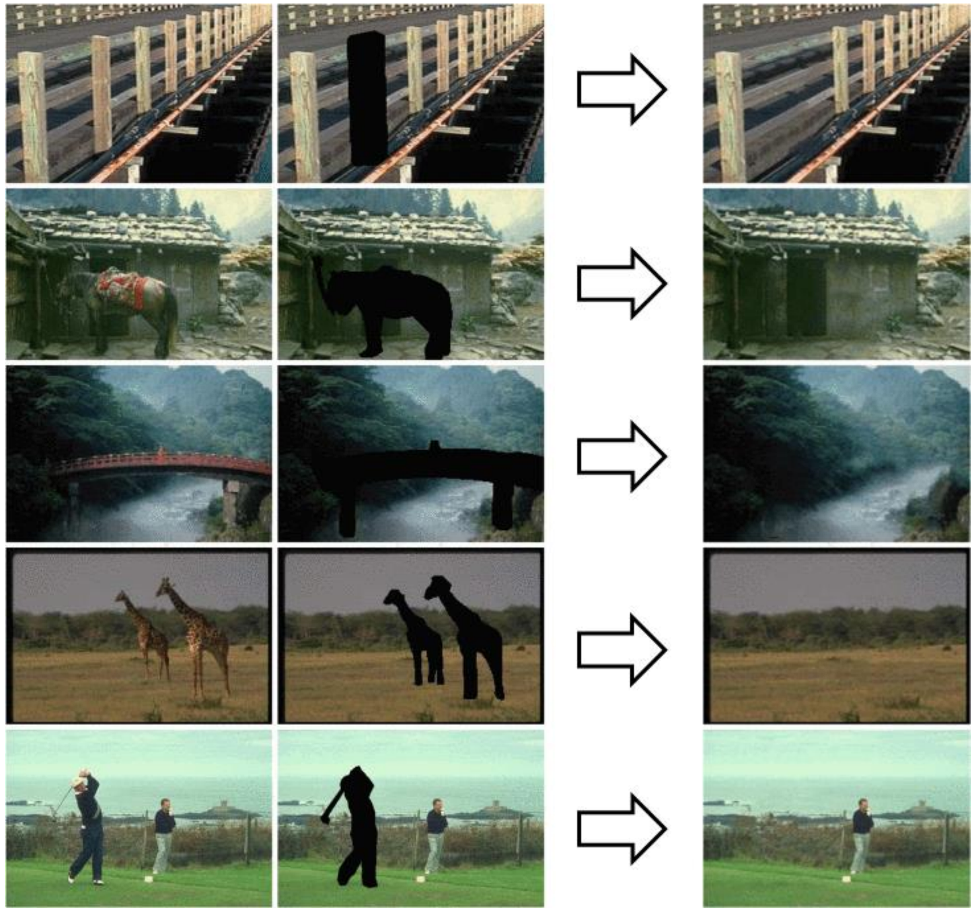
- Define bi-dimensional lattice on the image
  - Regular grid, patches, superpixels, segments
- Background/Foreground segmentation
  - X_i Observable label
  - Y_i Region annotation as background/foreground
- Impose constraints
  - $f_S(Y_i, X_i)$ $\Rightarrow$ Cost of disregarding available annotation
  - $f_H(Y_i, Y_j) \approx [y_i \neq y_j] w_{ij}$ $\Rightarrow$ Label affinity constraint weighted by region similarity w_{ij}



Background Segmentation

Background Segmentation

Image Completion



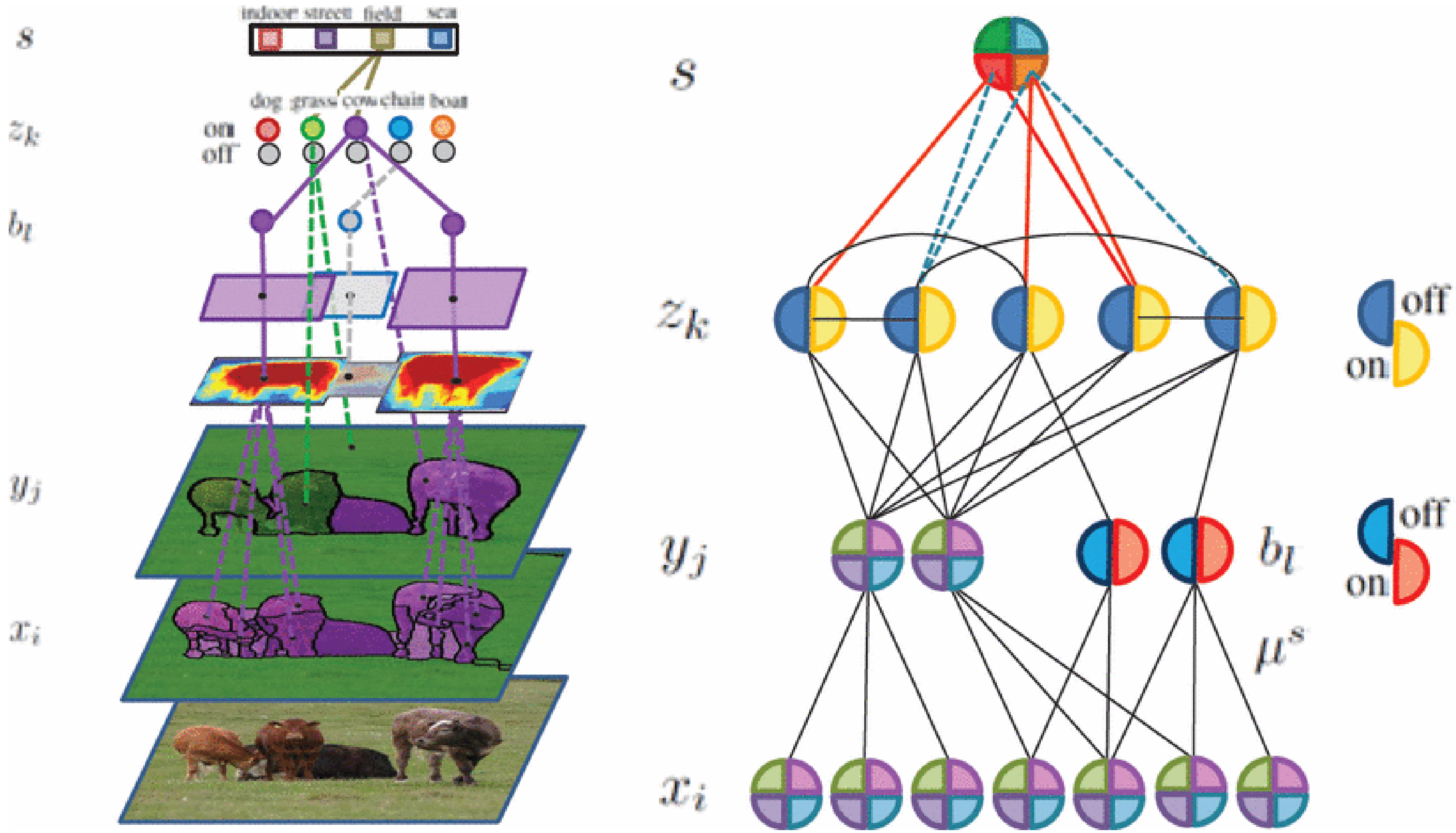
N. Komodakis. Image Completion Using Global Optimization. CVPR 2006

Image Completion

N. Komodakis. Image Completion Using Global Optimization. CVPR 2006

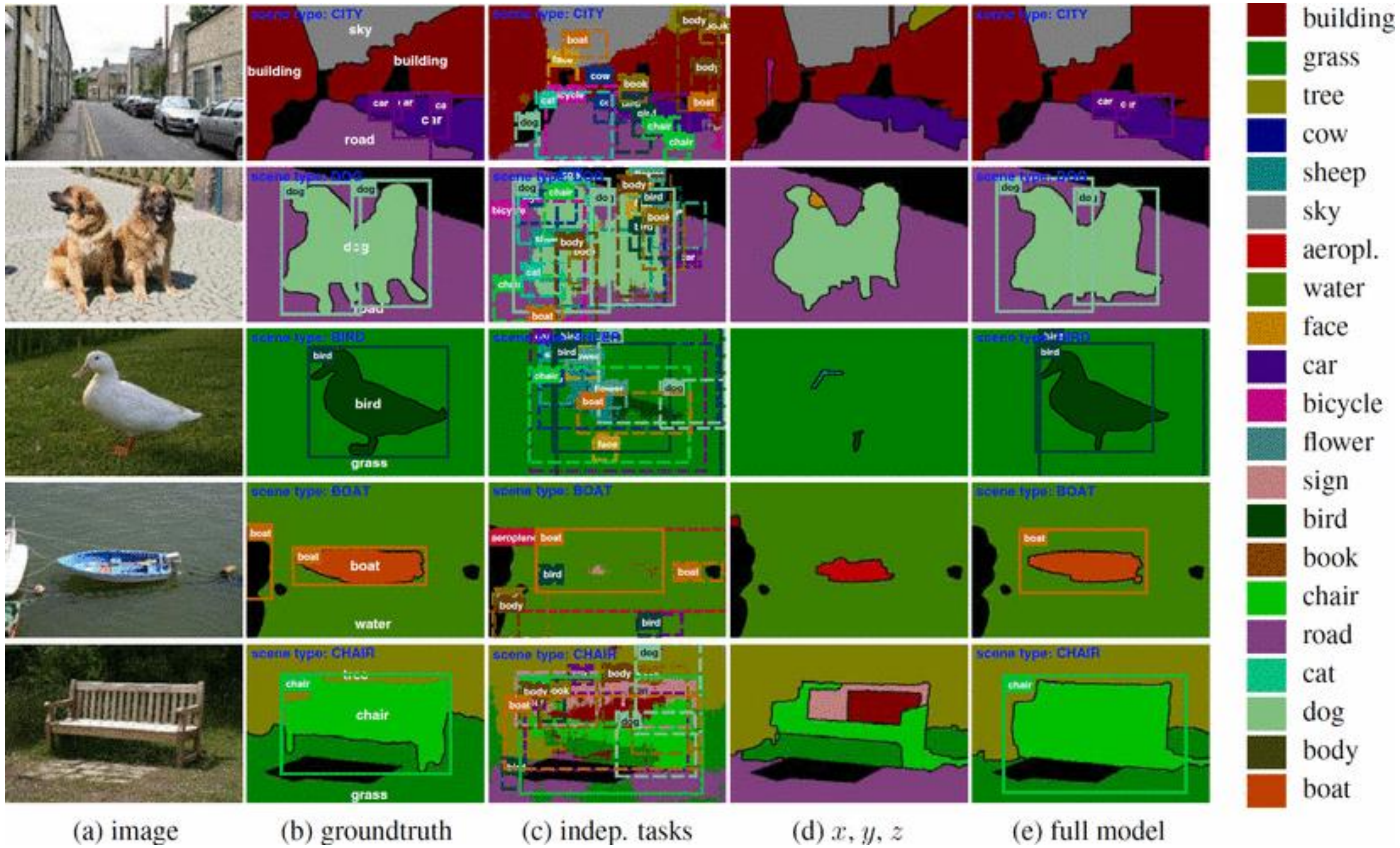
Università di Pisa

Semantic Segmentation



J. Yao, S. Fidler and R. Urtasun, "Describing the scene as a whole: Joint object detection, scene classification and semantic segmentation," ICCV 2012

Semantic Segmentation



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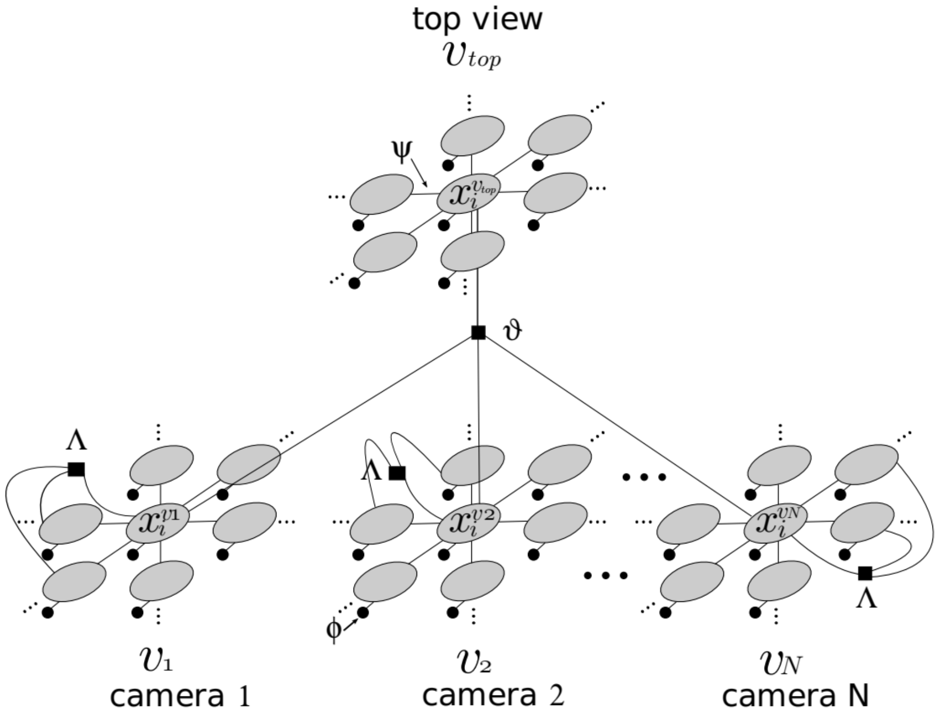
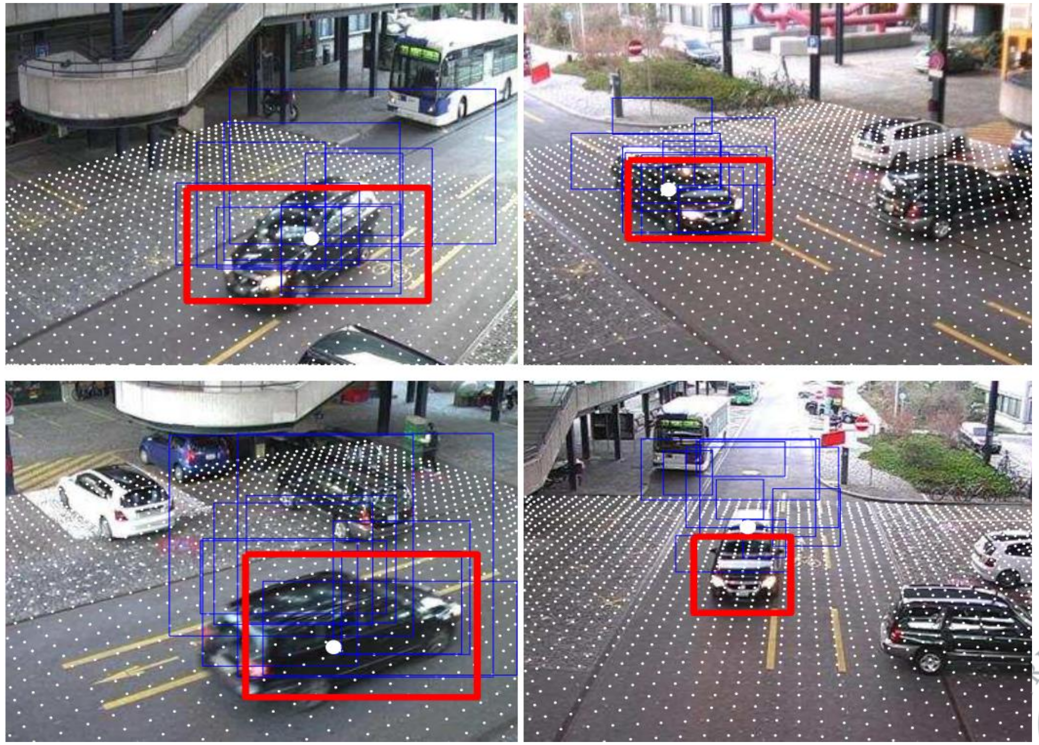
Integrating Prior Information





Roig et al "Conditional Random Fields for multi-camera object detection," ICCV 2011

MRF Software

- CRFsuite - Fast implementation of linear/chain CRFs for NLP applications (native C++; Scikit-like package python-crfsuite)
- PyStruct - Python CRF package including 2D lattices, graph structures and several inference algorithms
- pgmpy - Python library for graphical models (includes CRF, MRF and more)
- Pyro - Ubers' own PyTorch provide an implementation of Deep CRF
- UGM - Matlab library for Markov Random Fields
- CRF implementations (in particular linear) are present in major DL libraries (e.g. Tensorflow, PyTorch)

A Python Example

```
from pgmpy . models import MarkovModel
from pgmpy . factors . discrete import DiscreteFactor
import numpy as np
from pgmpy . inference import BeliefPropagation
. . .
MM=MarkovModel ( ) ;
# Add edges ( and nodes if not existent )
MM. add_edges_from ( [ ( 'f1', 'f2' ), ( 'f2', 'f3' ), ( 'o1', 'f1' ), ( 'o2', 'f2' ), ( 'o3', 'f3' ) ] )

#Generate transition feature
transition =np . array ( [ 10, 90, 90, 10 ] ) ;
#Generate corresponding factor
factorH1 = DiscreteFactor ( [ 'f1', 'f2' ], cardinality = [ 2, 2 ], values = transition )
#Add it to the model
MM. add _factors ( factorH1 )

#Solve smoothing by belief propagation ( i.e. estimate hidden RV)
belief_propagation = BeliefPropagation (MM)
ymax = belief_propagation . map_query ( variables = [ 'f1', 'f2', 'f3' ] , \
evidence = { 'o1' : toVal ( 'class1' ), 'o2' : toVal ( 'class1' ), 'o3' : toVal ( 'class2' ) } )
...
```

Take Home Messages

- Markov Random Fields
  - Undirected graphical models
  - Allow to express constraints between RV without needing to use probabilities
  - Topology follows data structure/relations and allow embedding prior information
- Conditional Random Fields
  - Constrained MRF learning discriminative posteriors
  - Feature functions to model constraints (often simple hand-coded feature detectors)
  - Parameters allow to linearly combine features
- CRF/MRF are often used as final refinement (segmentation, POS tagging, ...)

Next 2 Lectures

Bayesian Learning and Variational Inference

- Bayesian latent variable models
- Variational bound and its optimization
- Latent Dirichlet Allocation
  - Possibly the simplest Bayesian latent variable model
  - Variational Expectation-Maximization
  - Applications to machine vision