Convex sets and convex functions

Mauro Passacantando

Department of Computer Science, University of Pisa
mauro.passacantando@unipi.it

Numerical Methods and Optimization
Master in Computer Science – University of Pisa

Subspaces

Given $x, y \in \mathbb{R}^n$.

A linear combination of x and y is a point $\alpha x + \beta y$, where $\alpha, \beta \in \mathbb{R}$.

A set $C \subseteq \mathbb{R}^n$ is a subspace if it contains all the linear combinations of any two points in C.

Examples:

- $\{0\}$
- any line which passes through zero
- the solution set of a homogeneous system of linear equations

$$C = \{x \in \mathbb{R}^n : \; Ax = 0\},$$

where A is a $m \times n$ matrix.

Affine sets

An affine combination of x and y is a point $\alpha x + \beta y$, where $\alpha + \beta = 1$.

A set $C \subseteq \mathbb{R}^n$ is an affine set if it contains all the affine combinations of any two points in C.

Examples:

- any single point $\{x\}$
- any line
- the solution set of a system of linear equations

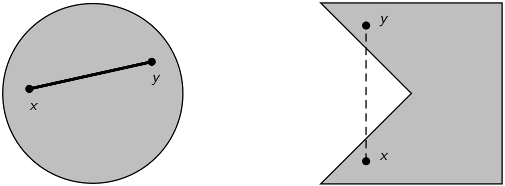
  $$C = \{x \in \mathbb{R}^n : \ Ax = b\},$$

  where A is a $m \times n$ matrix and $b \in \mathbb{R}^m$
- any subspace

Convex sets

A convex combination of two given points x and y is a point $\alpha x + \beta y$, where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta = 1$.

A set $C \subseteq \mathbb{R}^n$ is convex if it contains all the convex combinations of any two points in C.

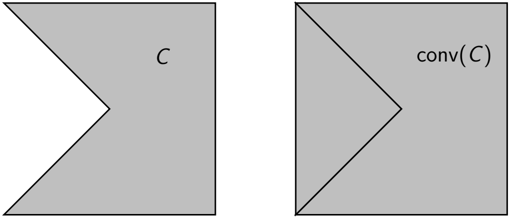


convex set non-convex set

Exercise. Prove that if C is convex, then for any $x^1, \dots, x^k \in C$ and $\alpha_1, \dots, \alpha_k \in [0, 1]$ s.t. $\sum_{i=1}^{k} \alpha_i = 1$, one has $\sum_{i=1}^{k} \alpha_i x^i \in C$.

Convex hull

The convex hull conv(C) of a set C is the smallest convex set containing C.



Exercise. Prove that conv(C) $= \{$all convex combinations of points in $C\}$.

Exercise. Prove that C is convex if and only if $C =$ conv(C).

Convex sets - Examples

Examples:

- subspace
- affine set
- line segment
- halfspace $\{x \in \mathbb{R}^n : a^\mathsf{T} x \leq b\}$
- polyhedron $P = \{x \in \mathbb{R}^n : \ Ax \leq b\}$ solution set of a system of linear inequalities
- Euclidean ball $B(x, r) = \{y \in \mathbb{R}^n : \|y - x\| < r\}$

Operations that preserve convexity

Sum and difference

If C_1 and C_2 are convex then $C_1 + C_2 := \{x + y : \; x \in C_1, \; y \in C_2\}$ is convex.
If C_1 and C_2 are convex then $C_1 - C_2 := \{x - y : \; x \in C_1, \; y \in C_2\}$ is convex.

Intersection

If C_1 and C_2 are convex then $C_1 \cap C_2$ is convex.
If $\{C_i\}_{i \in I}$ is a (possibly infinite) family of convex sets then $\bigcap_{i \in I} C_i$ is convex.

Union

If C_1 and C_2 are convex then $C_1 \cup C_2$ is convex?

Closure and interior

If C is convex then $\mathrm{cl}(C)$ is convex.
If C is convex then $\mathrm{int}(C)$ is convex.

Operations that preserve convexity

Affine functions

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be affine, i.e. $f(x) = Ax + b$, with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

- If $C \subseteq \mathbb{R}^n$ is convex, then $f(C) = \{f(x) : \ x \in C\}$ is convex
- If $C \subseteq \mathbb{R}^m$ is convex, then $f^{-1}(C) = \{x \in \mathbb{R}^n : \ f(x) \in C\}$ is convex

Examples:

- scaling, e.g. $f(x) = \alpha\, x$, with $\alpha > 0$
- translation, e.g. $f(x) = x + b$, with $b \in \mathbb{R}^n$
- rotation, e.g. $f(x) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} x$, with $\theta \in (0, 2\pi)$

Cones

A set $C \subseteq \mathbb{R}^n$ is a cone if $\alpha\, x \in C$ for any $x \in C$ and $\alpha \geq 0$.

Examples:

- $\mathbb{R}^n_+$ is a convex cone
- $\{x \in \mathbb{R}^2 : \; x_1\, x_2 = 0\}$ is a non-convex cone
- Given a polyhedron $P = \{x : \; Ax \leq b\}$, the recession cone of P is defined as

$$\text{rec}(P) := \{d : \; x + \alpha\, d \in P \quad \text{for any } x \in P,\ \alpha \geq 0\}.$$

  It is easy to prove $\text{rec}(P) = \{x : \; Ax \leq 0\}$, thus it is a polyhedral cone.
- $\{x \in \mathbb{R}^3 : \; x_3 \geq \sqrt{x_1^2 + x_2^2}\}$ is a non-polyhedral cone.

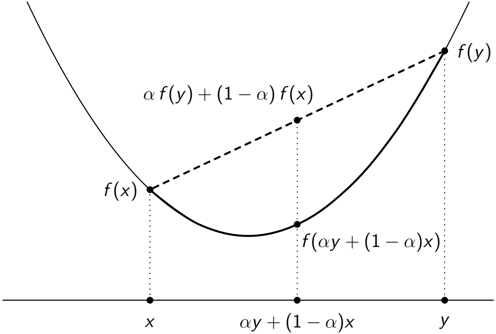
Exercises

1. Write the vector $(1,1)$ as the convex combination of the vectors $(0,0),(3,0),(0,2),(3,2)$.
2. When does one halfspace contain another? Give conditions under which
$$\{x \in \mathbb{R}^n : \; a_1^{\mathsf{T}} x \leq b_1\} \subseteq \{x \in \mathbb{R}^n : \; a_2^{\mathsf{T}} x \leq b_2\},$$
where $\|a_1\| = \|a_2\| = 1$. Also find the conditions under which the two halfspaces are equal.
3. Which of the following sets are polyhedra?
   a) $\{y_1 a_1 + y_2 a_2 : \; -1 \leq y_1 \leq 1, \; -1 \leq y_2 \leq 1\}$, where $a_1, a_2 \in \mathbb{R}^n$.
   b) $\left\{x \in \mathbb{R}^n : \; x \geq 0, \; \sum_{i=1}^n x_i = 1, \; \sum_{i=1}^n a_i x_i = b_1, \; \sum_{i=1}^n a_i^2 x_i = b_2\right\}$, where $b_1, b_2, a_1, \dots, a_n \in \mathbb{R}$.
   c) $\{x \in \mathbb{R}^n : \; x \geq 0, \; a^{\mathsf{T}} x \leq 1 \text{ for all } a \text{ with } \|a\| = 1\}$.

Convex functions

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\text{dom}(f)$ is convex and

$$f(\alpha y + (1-\alpha)x) \le \alpha f(y) + (1-\alpha) f(x) \qquad \forall\, x, y \in \text{dom}(f), \forall\, \alpha \in (0,1)$$



f is said concave if $-f$ is convex.

Exercise. Prove that if f is convex, then for any $x^1, \dots, x^k \in \text{dom}(f)$ and $\alpha_1, \dots, \alpha_k \in (0,1)$ s.t. $\sum_{i=1}^{k} \alpha_i = 1$, one has $f\left(\sum_{i=1}^{k} \alpha_i x^i\right) \le \sum_{i=1}^{k} \alpha_i f(x^i)$.

Strictly convex and strongly convex functions

$f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex if $\text{dom}(f)$ is convex and

$$f(\alpha y + (1-\alpha)x) < \alpha f(y) + (1-\alpha) f(x) \qquad \forall\, x, y \in \text{dom}(f), \alpha \in (0,1)$$

$f : \mathbb{R}^n \to \mathbb{R}$ is strongly convex if $\text{dom}(f)$ is convex and there exists $\tau > 0$ s.t.

$$f(\alpha y + (1-\alpha)x) \leq \alpha f(y) + (1-\alpha) f(x) - \frac{\tau}{2}\alpha(1-\alpha)\|y - x\|^2 \qquad \forall\, x, y \in \text{dom}(f), \alpha \in (0,1)$$

Thm. f is strongly convex if and only if $\exists\, \tau > 0$ s.t. $f(x) - \frac{\tau}{2}\|x\|^2$ is convex

Exercise.

- Prove that: strongly convex $\Longrightarrow$ strictly convex $\Longrightarrow$ convex
- convex $\Longrightarrow$ strictly convex ?
- strictly convex $\Longrightarrow$ strongly convex ?

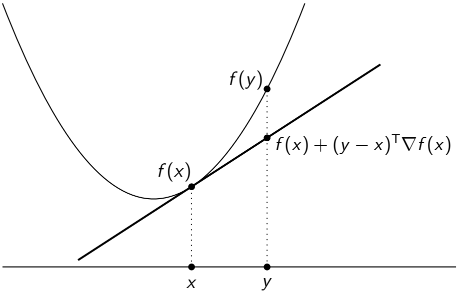
First order conditions

Assume f is continuously differentiable and $\text{dom}(f)$ is open.

Theorem

f is convex if and only if

$$f(y) \geq f(x) + (y-x)^{\mathsf{T}} \nabla f(x) \qquad \forall\, x, y \in \text{dom}(f).$$



First-order approximation of f is a global understimator

First order conditions

Theorem

- f is strictly convex if and only if

$$f(y) > f(x) + (y-x)^{\mathsf{T}} \nabla f(x) \qquad \forall\, x, y \in \mathrm{dom}(f), \text{with } x \neq y.$$

- f is strongly convex if and only if there exists $\tau > 0$ such that

$$f(y) \geq f(x) + (y-x)^{\mathsf{T}} \nabla f(x) + \frac{\tau}{2}\|y-x\|_2^2 \qquad \forall\, x, y \in \mathrm{dom}(f).$$

Second order conditions

Assume f is twice continuously differentiable and $\text{dom}(f)$ is open.

Theorem

- f is convex if and only if for all $x \in \text{dom}(f)$ the Hessian matrix $\nabla^2 f(x)$ is positive semidefinite, i.e.

$$v^\mathsf{T} \nabla^2 f(x) v \geq 0 \qquad \forall\ v \neq 0,$$

  or, equivalently, the eigenvalues of $\nabla^2 f(x)$ are ≥ 0.
- If $\nabla^2 f(x)$ is positive definite or, equivalently, the eigenvalues of $\nabla^2 f(x)$ are > 0 for all $x \in \text{dom}(f)$, then f is strictly convex.
- f is strongly convex if and only if there exists $\tau > 0$ such that $\nabla^2 f(x) - \tau I$ is positive semidefinite for all $x \in \text{dom}(f)$, i.e.

$$v^\mathsf{T} \nabla^2 f(x) v \geq \tau \|v\|_2^2 \qquad \forall\ v \neq 0,$$

  or, equivalently, the eigenvalues of $\nabla^2 f(x)$ are $\geq \tau$.

Examples

$f(x) = c^{\mathsf{T}} x$ is both convex and concave
$f(x) = \frac{1}{2} x^{\mathsf{T}} Q x + c^{\mathsf{T}} x$ is

- convex iff Q is positive semidefinite
- strongly convex iff Q is positive definite
- concave iff Q is negative semidefinite
- strongly concave iff Q is negative definite

$f(x) = e^{ax}$ for any $a \in \mathbb{R}$ is strictly convex, but not strongly convex
$f(x) = \log(x)$ is strictly concave, but not strongly concave
$f(x) = x^a$ for $a > 1$ or $a < 0$ is strictly convex. Is it strongly convex?
$f(x) = x^a$ for $0 < a < 1$ is strictly concave
$f(x) = \|x\|$ is convex, but not strictly convex
$f(x) = \max\{x_1, \ldots, x_n\}$ is convex, but not strictly convex

Exercises

1. Prove that the function
$$f(x_1, x_2) = \frac{x_1\, x_2}{x_1 - x_2}$$
is convex on the set $\{x \in \mathbb{R}^2 : \ x_1 - x_2 > 0\}$.
2. Prove that $f(x_1, x_2) = \dfrac{1}{x_1\, x_2}$ is convex on the set $\{x \in \mathbb{R}^2 : \ x_1, x_2 > 0\}$.
3. Given a convex set $C \subseteq \mathbb{R}^n$, the distance function is defined as follows:
$$d_C(x) = \inf_{y \in C} \|y - x\|.$$
Prove that d_C is a convex function.
4. Given $C = \{x \in \mathbb{R}^2 : \ x_1^2 + x_2^2 \leq 1\}$, write the distance function d_C explicitly.
5. Prove that the arithmetic mean of n positive numbers $x_1, \ldots, x_n$ is greater or equal to their geometric mean, i.e.,
$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$
(Hint: exploit the *log* function.)

Operations that preserve convexity

Theorem

- If f is convex and $\alpha > 0$, then αf is convex
- If f_1 and f_2 are convex, then $f_1 + f_2$ are convex
- If f is convex, then $f(Ax + b)$ is convex

Examples

- Log barrier for linear inequalities:

$$f(x) = -\sum_{i=1}^{m} log(b_i - a_i^{\mathsf{T}} x) \qquad \mathrm{dom}(f) = \{x : \; a_i^{\mathsf{T}} x < b_i\}$$

- Norm of affine function: $f(x) = \|Ax + b\|$

Exercise. If f_1 and f_2 are convex, then $f_1 \, f_2$ is convex?

Pointwise maximum

Theorem

- If $f_1, \ldots, f_m$ are convex, then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex.
- If $\{f_i\}_{i \in I}$ is a family of convex functions, then $f(x) = \sup_{i \in I} f_i(x)$ is convex.

Example. If $L(x, \lambda) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is convex in x and concave in λ, then

$$p(x) = \sup_{\lambda} L(x, \lambda) \quad \text{is convex}$$
$$d(\lambda) = \inf_{x} L(x, \lambda) \quad \text{is concave}$$

Composition

$f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$.

Theorem

- If f is convex and g is convex and nondecreasing, then $g \circ f$ is convex.
- If f is concave and g is convex and nonincreasing, then $g \circ f$ is convex.

- If f is concave and g is concave and nondecreasing, then $g \circ f$ is concave.
- If f is convex and g is concave and nonincreasing, then $g \circ f$ is concave.

Examples

- If f is convex, then $e^{f(x)}$ is convex
- If f is concave and positive, then $\log f(x)$ is concave
- If f is convex, then $-\log(-f(x))$ is convex on $\{x : \ f(x) < 0\}$
- If f is concave and positive, then $\dfrac{1}{f(x)}$ is convex
- If f is convex and nonnegative, then $f(x)^p$ is convex for all $p \geq 1$

Sublevel sets

Given $f : \mathbb{R}^n \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, the set

$$S_\alpha(f) = \{x \in \mathrm{dom}(f) : \ f(x) \leq \alpha\}$$

is said the α-sublevel set of f.

Exericise.

- If f is convex, then $S_\alpha(f)$ is a convex set for any $\alpha \in \mathbb{R}$.
- Is the converse true?

Exercise

Express each of the following convex sets

a) $\text{conv}\{(-1,-1),(1,0),(0,2)\}$

b) $\text{conv}\{(0,0),(1,1)\}$

c) $\text{conv}\left(\{x \in \mathbb{R}^2 : \ x_1^2 + (x_2 - 1)^2 = 1\} \cup \{x \in \mathbb{R}^2 : \ x_1^2 + (x_2 + 1)^2 = 1\}\right)$

d) $\text{conv}\{x \in \mathbb{R}^2 : \ x_1 x_2 = 1\}$

in the form $\bigcap_{i \in I} \{x : f_i(x) \leq 0\}$, where $f_i : \mathbb{R}^n \to \mathbb{R}$ are suitable convex functions.

Optimization problem in standard form

$$\begin{cases} \min \ f(x) \\ g(x) \leq 0 \\ h(x) = 0 \end{cases}$$

- $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function
- $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$ are the inequality constraints functions
- $h_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \dots, p$ are the equality constraints functions

Domain: $\mathcal{D} = \text{dom}(f) \cap \bigcap_{i=1}^{m} \text{dom}(g_i) \cap \bigcap_{j=1}^{p} \text{dom}(h_j)$

Feasible region: $\Omega = \{x \in \mathcal{D} : \ g(x) \leq 0, \ h(x) = 0\}$

implicit constraint: $x \in \mathcal{D}$
explicit constraints: $g(x) \leq 0$, $h(x) = 0$

Global and local optima

Global minimum: a feasible point x^* s.t. $f(x^*) \leq f(x)$ for all $x \in \Omega$

Local minimum: a feasible point x^* s.t. $f(x^*) \leq f(x)$ for all $x \in \Omega \cap B(x^*, R)$ for some $R > 0$

Optimal value:

$$v^* = \inf\{f(x) : \ x \in \Omega\}$$

$v^* = -\infty$ if the problem is unbounded below
$v^* = +\infty$ if the problem is infeasible

Examples

- $f(x) = \log(x)$, $v^* = -\infty$, no global minimum
- $f(x) = e^x$, $v^* = 0$, no global minimum
- $f(x) = x \log(x)$, $v^* = -1/e$, $x^* = 1/e$ is global minimum
- $f(x) = x^3 - 3x$, $v^* = -\infty$, $x^* = 1$ is local minimum

Convex optimization problems

An optimization problem

$$\begin{cases} \min \ f(x) \\ g(x) \leq 0 \\ h(x) = 0 \end{cases}$$

is said convex if f is convex, $g_1, \dots, g_m$ are convex and $h_1, \dots, h_p$ are affine.

Examples

a) $\begin{cases} \min \ x_1^2 + x_1 x_2 + 3x_2^2 + 4x_1 + 5x_2 \\ x_1^2 + x_2^2 - 4 \leq 0 \\ x_1 + x_2 - 2 = 0 \end{cases}$ is convex

b) $\begin{cases} \min \ x_1^2 + x_2^2 \\ x_1/(1 + x_2^2) \leq 0 \\ (x_1 + x_2)^2 = 0 \end{cases}$ is NOT convex, but it is equivalent to:

$$\begin{cases} \min \ x_1^2 + x_2^2 \\ x_1 \leq 0 \\ x_1 + x_2 = 0 \end{cases}$$ which is convex

Convex optimization problems

Why convex problems are important?

Theorem 1

In a convex optimization problem the feasible region is convex.

Theorem 2

In a convex optimization problem any local minimum is a global minimum.

Proof. Let x^* be a local minimum, i.e., there is $R > 0$ s.t.

$$f(x^*) \leq f(z) \qquad \forall\, z \in \Omega \cap B(x^*, R).$$

By contradiction, assume that x^* is not a global minimum, i.e., there is $y \in \Omega$ s.t. $f(y) < f(x^*)$. Take $\alpha \in (0, 1)$ s.t. $\alpha x^* + (1 - \alpha)y \in B(x^*, R)$. Then we have

$$f(x^*) \leq f(\alpha x^* + (1-\alpha)y) \leq \alpha f(x^*) + (1-\alpha)f(y) < f(x^*),$$

which is impossible. □