

Exercises on Constrained Optimization

Answer the following questions for any of the nonlinear optimization problems reported below:

- Is it a convex optimization problem?
- Do global optimal solutions exist? Why?
- Which constraints qualification holds?
- Find all the solutions of the KKT system associated with the problem.
- Find local and global optimal solutions exploiting the optimality conditions.
- Write the Lagrangian dual problem and try to solve it.

$$1. \begin{cases} \min & -x_1 + x_2^2 \\ & -x_1^2 - x_2^2 + 4 \leq 0 \end{cases}$$

$$2. \begin{cases} \min & x_1^3 + x_2^3 \\ & -x_1 - 1 \leq 0 \\ & -x_2 - 1 \leq 0 \end{cases}$$

$$3. \begin{cases} \min & -x_1^2 - x_2^2 \\ & x_1^2 + x_2^2 - 1 \leq 0 \end{cases}$$

$$4. \begin{cases} \min & -3x_1^2 - 3x_2^2 + 4x_1x_2 \\ & -x_1 - x_2 + 2 \leq 0 \\ & x_1^2 + x_2^2 - 4 \leq 0 \end{cases}$$

$$5. \begin{cases} \min & x_2 - x_1^2 \\ & x_1^2 + x_2^2 - 4 \leq 0 \\ & x_2 \leq 0 \end{cases}$$

$$6. \begin{cases} \min & 4x_1^2 + x_2^2 - x_1x_2 - 14x_1 - 2x_2 \\ & -x_1 \leq 0 \\ & -x_2 \leq 0 \\ & x_1 + x_2 - 1 \leq 0 \end{cases}$$

$$7. \begin{cases} \min & -(x_1 - 3)^2 - 4(x_2 - 2)^2 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{cases}$$

$$8. \begin{cases} \min & -3x_1^2 - 3x_2^2 + 4x_1x_2 \\ & -x_1 - x_2 + 2 \leq 0 \\ & x_1^2 + x_2^2 - 4 \leq 0 \end{cases}$$

$$9. \begin{cases} \min & -x_1^2 - 2x_2^2 \\ & -x_1 + 1 \leq 0 \\ & -x_2 + 1 \leq 0 \\ & x_1 + x_2 - 6 \leq 0 \end{cases}$$

10. Find the largest circle contained in the triangle

$$T := \{x \in \mathbb{R}^2 : x_1 \geq 0, \quad x_2 \geq 0, \quad 3x_1 + 4x_2 \leq 12\}.$$

11. Find the distance between the circle $C = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ and the line $r = \{x \in \mathbb{R}^2 : x_1 - x_2 = 2\}$.

12. Given the line

$$r = \{x \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 4, \quad x_1 + x_2 + x_3 = 8\},$$

find the point in r closest to the origin.

13. Find the projection of a point $z \in \mathbb{R}^n$ on the ball with center a and radius R .

14. Find the projection of a point $z \in \mathbb{R}^n$ on the box $\{x \in \mathbb{R}^n : \ell_i \leq x_i \leq u_i\}$.

15. Find the distance between two parallel hyperplanes

$$H_1 = \{x \in \mathbb{R}^n : a^\top x = b_1\}, \quad H_2 = \{x \in \mathbb{R}^n : a^\top x = b_2\}, \quad b_1 \neq b_2.$$

16. Find the distance between two balls

$$B_1 = \{x \in \mathbb{R}^n : \|x - a\| \leq R_1\}, \quad B_2 = \{x \in \mathbb{R}^n : \|x - b\| \leq R_2\}.$$