Exam

Numerical Methods and Optimization course
University of Pisa, 2017-01-16

You may use Matlab, pencil or paper, or a calculator (unless explicitly stated in the exercise). You may use the quick reference sheet on Matlab's syntax posted on the web page of the course.

Exercise 1. Let $A \in \mathbb{C}^{m \times n}$, with $m > n$, and $B = AA^*$.

a. How are the eigenvalues of B related to the singular values of A?

b. Write a Matlab `function [v, lambda] = powersB(A, k)` that performs k iterations of the power method on B (with normalization), starting from $x_0 = e_1$ (where e_1 is the first column of the identity matrix). Be careful with the products: your implementation should require $O(mn)$ operations per step, not $O(m^2)$.

c. Generate a matrix with the commands

```
rng(0); A = randn(100, 4);
```

What is the value of the residual $\frac{\|Bv - \lambda_1 v\|}{\|v\|}$ obtained after 200 iterations of the method you have implemented? What is the computed eigenvalue λ_1?

d. Let $A = Q_1 R_1$ be the thin QR factorization of A, and let $w \in \mathbb{C}^n$ be an eigenvector of the $n \times n$ matrix $R_1 R_1^*$. Show that $Q_1 w$ is an eigenvector of B.

e. We want to use the relation found at the previous item to compute eigenvectors of B. Write a Matlab `function [v, lambda] = inverseB(A, k, mu)` that computes the thin QR factorization of A (using Matlab's builtin function `qr(A, 0)`), then performs k steps of inverse iteration (with shift μ) on $R_1 R_1^*$, and finally uses the computed approximation $\tilde{w} \in \mathbb{C}^n$ to construct an eigenvector v of B.

f. What is the value of λ_2 produced by `[v2, lambda2] = inverseB(A, 15, 100)`?

(You can compute the eigenvalues of B explicitly (using Matlab's `eig(A*A')`*) and use them to check your results.)*

Exercise 2. Consider the following constrained optimization problem:

$$\begin{cases} \min\ -x_1^2 - x_2^2 - 6x_1 - 4x_2 \\ x_1 \leq 0 \\ x_2 \leq 0 \\ -x_1 - x_2 \leq 2 \end{cases}$$

a. Do global optimal solutions exist? Why?

b. Is it a convex problem? Why?

c. Do constraint qualifications hold in any feasible point?

d. Is the point $(-2, 0)$ a local minimum? Why?

e. Find all the solutions of the KKT system.

f. Find local minima and global minima.

Exercise 3. Consider the following unconstrained optimization problem:

$$\begin{cases} \min\ x_1^2 + \frac{3}{2}x_2^2 + x_3^2 + \frac{3}{2}x_4^2 - x_1x_3 - 2x_2x_4 + x_1 + 2x_2 + 3x_3 + 4x_4 \\ x \in \mathbb{R}^4 \end{cases}$$

a. Is it a convex problem? Why?

b. Do global minima exist? Why?

c. Is the global minimum unique? Why?

d. Solve the problem by means of the gradient method with exact line search starting from the point $(0,0,0,0)$ and using $\|\nabla f(x)\| < 10^{-6}$ as stopping criterion. Which is the global minimum? Which is the optimal value? How many iterations are needed?

e. Solve the problem by means of the gradient method with inexact line search setting $\alpha = 0.5$, $\gamma = 0.9$, $\bar{t} = 1$, starting from the point $(0,0,0,0)$ and using $\|\nabla f(x)\| < 10^{-6}$ as stopping criterion. Which is the global minimum? How many iterations are needed?

f. Solve the problem by means of the conjugate gradient method starting from the point $(0,0,0,0)$ and using $\|\nabla f(x)\| < 10^{-6}$ as stopping criterion. Which is the global minimum solution? Write the vector x found at each iteration.

Solutions

Exercise 1.

a. Let $A = USV^*$ be an SVD of A. We have $B = USS^*U^*$, which is an eigenvalue decomposition because U is unitary and SS^* is a diagonal matrix with

$$SS^* = \begin{bmatrix} \sigma_1^2 & & & & & & & \\ & \sigma_2^2 & & & & & & \\ & & \ddots & & & & & \\ & & & \sigma_n^2 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{bmatrix}.$$

The eigenvalues of B are the diagonal entries of SS^*, which are the squares of the n singular values of A and $m - n$ zeros.

b.
```
function [x, lambda] = powersB(A, k);
% power method on B=AA^*

z = eye(size(A,1), 1);
for i = 1:k
    x = A * (A' * z);
    z = x / norm(x);
end
% the following two lines compute the Rayleigh quotient
% lambda = (z'*A*A'*z) / (z'*z) with lower cost
% (note that z'*z = norm(z)^2 = 1)
h = A' * z;
lambda = h' * h;
```

Note that forming B explicitly as `B = A*A'` or implicitly as `x = (A*A')*z` would cost $O(m^2n)$ operations.

c. With A and k as given, on my version of Octave I obtain `lambda = 124.61` and `norm(A*(A'*v) - lambda*v) / norm(v) = 1.1362e-13`. Different ways to put the parentheses can give slightly different residuals, all about `1e-13`.

d. Let w be such that $R_1R_1^*w = \lambda w$. Transposing $A = Q_1R_1$ we obtain $A^* = R_1^*Q^*$. We can compute directly

$$B(Q_1w) = AA^*Q_1w = Q_1R_1R_1^* \underbrace{Q_1^*Q_1}_{=I_n} w = Q_1 \underbrace{R_1R_1^*w}_{=\lambda w} = \lambda Q_1 w.$$

e.
```
function [v, lambda] = inverseB(A, k, mu)
% inverse power method on B=AA^*

[m, n] = size(A);
[Q1, R1] = qr(A, 0);

[L, U] = lu(R1*R1'-mu*eye(n));
z = eye(n, 1);
for i = 1:k
    x = U\(L\z);
    z = x / norm(x);
end
v = Q1 * z;
h = A' * v;
lambda = h' * h;
```

Either a QR and LU factorization of $R_1 R_1^* - \mu I$ can be used. The matrix is Hermitian, but not positive definite in general, so we cannot use Cholesky. (There is actually a factorization tailored to Hermitian indefinite matrices, the LDL factorization, but we did not see it during the course.)

f. With my version of Octave I obtain `lambda2 = 105.09`.

Exercise 2.

a. Yes, the objective function is continuous and the feasible region is closed and bounded (Weierstrass Theorem).

b. No, the objective function is not convex since $\nabla^2 f(x) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$.

c. Yes, the constraints are linear.

d. No, since it does not solve the KKT system.

e. The solutions of the KKT system are $x = (0, -2)$ with $\lambda = (6, 0, 0)$ and $x = (0, 0)$ with $\lambda = (6, 4, 0)$.

f. $(0, 0)$ is the global minimum, $(0, -2)$ is not a local minimum.

Exercise 3.

a. Yes, the hessian matrix of the objective function f is

$$Q = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 3 & 0 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 3 \end{pmatrix}$$

The eigenvalues of Q are 1, 1, 3, 5, hence f is strongly convex.

b. Yes, since f is strongly convex.

c. Yes, since f is strongly convex.

d. After 34 iterations the algorithm finds the approximated global minimum $x = (-1.6667, -2.8000, -2.3333, -3.2000)$ with value -13.5333.

e. After 24 iterations the algorithm finds the approximated global minimum $x = (-1.6667, -2.8000, -2.3333, -3.2000)$.

f. The algorithm stops after 3 iterations.
Iteration 1: $x = (-0.7143, -1.4286, -2.1429, -2.8571)$,
iteration 2: $x = (-1.3949, -2.7898, -2.5032, -3.0573)$,
iteration 3: $x = (-1.6667, -2.8000, -2.3333, -3.2000)$.