Exam

Numerical Methods and Optimization
University of Pisa, 2017-04-11

You may use Matlab, pencil or paper, or a calculator (unless explicitly stated in the exercise). You may use the quick reference sheet on Matlab's syntax posted on the web page of the course.

Exercise 1. Let $A = QTQ^*$ be the Schur decomposition of a matrix $A \in \mathbb{C}^{n\times n}$.

a. What is the Schur decomposition of the matrix $A - \tau I$, for $\tau \in \mathbb{C}$?

b. Show how, given the matrices Q, T, one can solve a linear system $(A - \tau I)x = b$ in time $O(n^2)$.

c. Write a `function x = inverse_iteration(Q, T, k, tau)` that performs `k` steps of inverse power iteration with shift `tau` on the matrix A, given the factors Q, T of its Schur decomposition (and returns the current approximation of an eigenvector). As a starting vector, use `x0 = ones(n, 1)`. Write the source code of this function on paper in your solution.

   Your function should run in time $O(n^2k)$.

d. Use your function (with $k = 20$ iterations) to compute the smallest eigenvalue of the matrix $A = QTQ^*$ defined by

$$Q = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}, \quad T = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

   What is the returned vector x? What is the value of $\|Ax - x\|_2$?

e. Suppose that you have a matrix A for which you know the Schur decomposition $A = QTQ^*$. Would you recommend the above function as a method to compute its eigenvalues? Is it an effective strategy?

Exercise 2. Consider the following constrained optimization problem:

$$\begin{cases} \min \ -x_1^2 - x_2^2 + 4x_1 - 2x_2 \\ x_2 \leq 0 \\ x_1^2 + x_2^2 - 4x_1 \leq 0 \end{cases}$$

a. Do global optimal solutions exist? Why?

b. Is the objective function convex? Why?

c. Do constraints qualifications hold in any feasible point? (For instance, the Slater constraints qualification).

d. Is the point $(2, -1)$ a local minimum? Why?

e. Find solutions of the KKT system.

f. Find global minima of the problem.

Exercise 3. Consider the following unconstrained optimization problem:

$$\begin{cases} \min \ 2x_1^2 + x_2^2 + x_3^2 + 2x_4^2 - x_1x_3 + x_3x_4 - 2x_1 - 4x_2 - 4x_3 - 6x_4 \\ x \in \mathbb{R}^4 \end{cases}$$

a. Is the objective function convex? Why?

b. Do global minima exist? Why?

c. Is the global minimum unique? Why?

d. Describe the gradient method with exact line search for solving the problem.

e. Describe a gradient method with inexact line search for solving the problem.

f. Describe the conjugate gradient method for solving the problem. What is the maximum number of iterations needed for finding the global minimum?

Solutions

Exercise 1.

a. $A - \tau I = QTQ^* - \tau I = QTQ^* - \tau QQ^* = Q(T - \tau I)Q^*$. So the Schur decomposition of $A - \tau I$ has the same Q and $T' = T - \tau I$.

b. One has

$$x = (A - \tau I)^{-1} b = (Q(T - \tau I)Q^*)^{-1} b = Q(T - \tau I)^{-1} Q^* b = Q((T - \tau I)^{-1}(Q^* b)).$$

So a procedure to solve the linear system is:

(a) Compute $c = Q^* b$;

(b) Solve the system $(T - \tau I)y = c$ by back-substitution (Matlab: `y = (T-tau*eye(n)) \ c`);

(c) Compute $x = Qy$.

c. A possible implementation is as follows.

```
function x = inverse_iteration(Q, T, k, tau)
n = length(Q);
x = ones(n, 1);
TT = T - tau*eye(n);
for i = 1:k
    x = Q * (TT \ (Q' * x));
    x = x / norm(x);
end
```

Note that `TT \ c` takes $O(n^2)$ because `TT` is upper triangular, and the two matrix-vector multiplications take $O(n^2)$ as well.

d. To compute the smallest eigenvalue of A (in modulus), we select the shift $\tau = 0$ and run the inverse iteration with `x = inverse_iteration(Q, T, 20, 0)`. The given function returns

```
>> x = inverse_iteration(Q, T, 20, 0)
x =
   -0.8944
    0.0000
   -0.4472
>> norm(Q*T*Q'*x-x)
ans =
   2.5605e-06
```

If needed, we can compute λ with the Rayleigh quotient $\lambda = \frac{x^* Q T Q^* x}{x^* x}$.

e. The eigenvalues of A are already explicitly available from the Schur form: they are the diagonal entries of T (since T and $A = QTQ^*$ are similar). So there is no need to do any operations to compute them, we can simply read them off from T.

Exercise 2.

a. Yes, the objective function is continuous and the feasible region is closed and bounded (Weierstrass's Theorem).

b. No, the Hessian matrix has negative eigenvalues, hence it is not positive semidefinite.

c. Yes, Slater condition holds: the constraints are convex and there is a feasible point, e.g. $(1, -1)$, such that no constraint is active.

d. No, $\nabla f(2, -1) = 0$ and $\nabla^2 f(2, -1)$ is negative definite, hence $(2, -1)$ is a local maximum.

e. The solutions of the KKT system are

   $x = (0, 0)$ $\lambda = (2, 1)$, $f(x) = 0$
   $x = (2, 0)$ $\lambda = (2, 0)$, $f(x) = 4$
   $x = (2, -1)$ $\lambda = (0, 0)$, $f(x) = 5$
   $x = (4, 0)$ $\lambda = (2, 1)$, $f(x) = 0$
   $x = (2, -2)$ $\lambda = (0, 1/2)$, $f(x) = 4$

f. The global minima are $(0, 0)$ and $(4, 0)$.

Exercise 3.

a. Yes, the eigenvalues of the matrix $\nabla^2 f$ are 1.2679, 2, 4 and 4.7321, hence f is strongly convex.

b. Yes, because f is strongly convex.

c. Yes, because f is strongly convex.

d. The gradient method with exact line search finds the global minimum $(1, 2, 2, 1)$ after 27 iterations (starting from $(0, 0, 0, 0)$ with stopping criterion $\|\nabla f(x)\| < 10^{-6}$).

e. The gradient method with inexact line search, where $\alpha = 0.1$, $\gamma = 0.9$ and $\bar{t} = 1$, finds the global minimum $(1, 2, 2, 1)$ after 39 iterations (starting from $(0, 0, 0, 0)$ with stopping criterion $\|\nabla f(x)\| < 10^{-6}$).

f. The conjugate gradient method finds the global minimum $(1, 2, 2, 1)$ after 4 iterations (starting from $(0, 0, 0, 0)$ with stopping criterion $\|\nabla f(x)\| < 10^{-6}$).