Exam

Numerical Methods and Optimization course
University of Pisa, 2017-06-13

You may use Matlab, pencil or paper, or a calculator (unless explicitly stated in the exercise). You may use the quick reference sheet on Matlab's syntax posted on the web page of the course.

Exercise 1. Let $A = QR$ be the QR factorization of a square matrix $A \in \mathbb{C}^{n\times n}$.

a. Show that given Q, R one can solve the linear system $(A^*A)x = b$ in time $O(n^2)$. (Recall that A^* is the conjugate transpose of A.)

b. Write a Matlab `function x = solve_system(Q, R, b)` that solves this linear system in time $O(n^2)$ as described above. (*You may use Matlab's operator \ to solve triangular linear systems.*)

c. Write a Matlab `function z = inverse_power(Q, R, k)` that performs k iterations of the inverse power method on the matrix A^*A, starting from the initial vector `z=ones(n, 1)`, and returns the approximate eigenvector $z \in \mathbb{C}^n$ (normalized such that $\|z\|_2 = 1$).

   If you have completed the previous points, try to use the same ideas to obtain cost $O(n^2)$ per iteration.

d. What is the value of z returned by the function `inverse_power` on the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix},$$

   with $k = 10$ steps?

e. How is this eigenvector z related to the SVD of A?

In points b and c, please report on paper the text of the function.

Exercise 2. Consider the following constrained optimization problem:

$$\begin{cases} \min \ x_1 + x_2 \\ x_1^2 + x_2^2 - 5 \leq 0 \\ x_1^2 - 1 \leq 0 \end{cases}$$

a. Do global optimal solutions exist? Why?

b. Is it a convex problem? Why?

c. Do constraint qualifications hold in any feasible point?

d. Is the point $(1, 2)$ a local minimum? Why?

e. Find all the solutions of the KKT system.

f. Find local minima and global minima.

g. Find the objective function and constraints of the Lagrangian dual problem.

h. Is (1, 0) an optimal solution of the Lagrangian dual problem? Why?

Exercise 3. Consider the following unconstrained optimization problem:

$$\begin{cases} \min \ 2x_1^2 + x_2^2 + 2x_3^2 + x_4^2 + x_1x_2 + x_1x_3 + 2x_3x_4 - x_1 + 4x_2 + 4x_3 - 2x_4 \\ x \in \mathbb{R}^4 \end{cases}$$

a. Is it a convex problem? Why?

b. Do global minima exist? Why?

c. Is the global minimum unique? Why?

d. Solve the problem by means of the gradient method with exact line search starting from the point $(0, 0, 0, 0)$ and using $\|\nabla f(x)\| < 10^{-6}$ as stopping criterion. What is the global minimum? What is the optimal value? How many iterations are needed?

e. Solve the problem by means of the gradient method with inexact line search setting $\alpha = 0.5$, $\gamma = 0.9$, $\bar{t} = 1$, starting from the point $(0, 0, 0, 0)$ and using $\|\nabla f(x)\| < 10^{-6}$ as stopping criterion. What is the global minimum? How many iterations are needed?

f. Solve the problem by means of the conjugate gradient method starting from the point $(0, 0, 0, 0)$ and using $\|\nabla f(x)\| < 10^{-6}$ as stopping criterion. What is the global minimum? Write the vector x found at each iteration.

Solutions

Exercise 1.

a. As seen in the lectures, we have $A^*A = (QR^*)QR = R^*Q^*QR = R^*R$. Hence

$$x = (R^*R)^{-1}b = R^{-1}(R^*)^{-1}b,$$

which we may solve with `w = R' \ b; x = R \ w`. Both systems are triangular, so the backslash operator performs back-substitution and requires $O(n^2)$ operations.

b.
```
function x = solve_system(Q, R, b)
w = R' \ b;
x = R \ w;
```

```
function z = inverse_power(Q, R, k)
n = size(R, 1);
z = ones(n, 1);
for it = 1:k
    z = R \ (R' \ z);
    z = z / norm(z);
end
```

d.
```
>> [Q,R] = qr(A); inverse_power(Q, R, 10)
ans =
    0.5833
   -0.7634
    0.2775
```

e. The vector z is the last column of V (up to a sign). Indeed, the eigenvalue decomposition of A^*A is $(USV)^*(USV) = V^*S^2V$, so the columns of V are the eigenvectors of A^*A, and its eigenvalues are the squares of the singular values σ_i. The inverse power method converges to the eigenvector relative to the smallest eigenvalue in modulus, which is the last one since the σ_i are sorted decreasingly.

Exercise 2.

a. Yes, the objective function is continuous and the feasible region is closed and bounded (Weierstrass Theorem).

b. Yes, the objective function and the constraints are convex.

c. Yes, the Slater condition holds.

d. No, since it does not solve the KKT system.

e. The solution of the KKT system is $x = (-1, -2)$ with $\lambda = (1/4, 1/4)$.

f. $(-1, -2)$ is the global minimum.

g. The Lagrangian dual problem is

$$\begin{cases} \max \ -\dfrac{1}{4(\lambda_1 + \lambda_2)} - \dfrac{1}{4\lambda_1} - 5\lambda_1 - \lambda_2 \\ \lambda_1 > 0 \\ \lambda_2 \geq 0 \end{cases}$$

h. No, the optimal value of the dual is -3, while $\varphi(1, 0) = -5.5$.

Exercise 3.

a. Yes, the hessian matrix of the objective function f is

$$Q = \begin{pmatrix} 4 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 4 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

The eigenvalues of Q are $0.6511, 1.6041, 4, 5.7448$ hence f is strongly convex.

b. Yes, since f is strongly convex.

c. Yes, since f is strongly convex.

d. After 48 iterations the algorithm finds the approximated global minimum $x = (2, -3, -4, 5)$ with value -20.

e. After 30 iterations the algorithm finds the approximated global minimum $x = (2, -3, -4, 5)$.

f. The algorithm stops after 3 iterations.
Iteration 1: $x = (0.6167, -2.4667, -2.4667, 1.2333)$
Iteration 2: $x = (2.1127, -3.7799, -2.9676, 3.4131)$
Iteration 3: $x = (2.0000, -3.0000, -4.0000, 5.0000)$.