Exam

Numerical Methods and Optimization course
University of Pisa, 2017-07-03

You may use Matlab, pencil or paper, or a calculator (unless explicitly stated in the exercise). You may use the quick reference sheet on Matlab's syntax posted on the web page of the course.

Exercise 1. Let $A = Q_1 R_1$ (with $Q_1 \in \mathbb{C}^{m\times n}$, $R_1 \in \mathbb{C}^{n\times n}$) be the thin QR factorization of a matrix $A \in \mathbb{C}^{m\times n}$, $m > n$.

a. Show that the pseudoinverse of A is equal to $A^+ = R_1^{-1} Q_1^*$.

b. Write a Matlab `function [x1, x2] = double_ls(A, b)` which returns the solution of the least squares problem $\min\|Ax - b\|$ computed with the method of normal equations (returned as x_1), and the same solution computed by multiplying b with the pseudoinverse A^+ obtained with the thin QR factorization as described above (returned as x_2).
(Transcribe on paper the code of this function.)

c. Let

$$A = \begin{bmatrix} 10 & 1 \\ 10 & 1 \\ 10 & 1 \\ 10 & 1.001 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1.999 \end{bmatrix}.$$

Show that, in exact arithmetic, the solution of the least squares problem $\min\|Ax - b\|$ is $\hat{x} = \left[\begin{smallmatrix} 0.3 \\ -1 \end{smallmatrix}\right]$.

d. What is the value of the differences $\|\hat{x} - x_1\|$, $\|\hat{x} - x_2\|$? Which is the most accurate method of the two, based on these values?

e. We wish to compare different ways to compute the product $x = R_1^{-1} Q_1^* b$. Suppose that we are given an exact expression for $S = R^{-1}$. Which is the computational cost (as a function of the dimensions m, n) of:

   i. computing $A^+ = SQ_1^*$ and then $x = A^+ b$;

   ii. computing $c = Q_1^* b$ and then $x = Sc$?

Exercise 2. Consider the following constrained optimization problem:

$$\begin{cases} \min \ -4x_1^2 - x_2^2 + 8x_1 - 2x_2 \\ x_1^2 - 2x_1 \leq 0 \\ x_2^2 - 2x_2 \leq 0 \end{cases}$$

a. Do global optimal solutions exist? Why?

b. Is it a convex problem? Why?

c. Does the constraints qualification hold in any feasible point? Why?

d. Is the point $x = (1, 0)$ a local minimum? Why?

e. Find all the solutions of the KKT system.

f. Find local minima and global minima.

g. Find the objective function and constraints of the Lagrangian dual problem.

h. Is $\lambda = (2, 4)$ an optimal solution of the Lagrangian dual problem? Why?

Exercise 3. Consider the following constrained optimization problem:

$$\begin{cases} \min \ x_1^2 + 2x_2^2 + 2x_3^2 + x_4^2 - x_1x_2 - x_1x_4 + 2x_2x_3 + 5x_1 + 5x_2 + 5x_3 + 5x_4 \\ x_1 + 2x_2 + 2x_3 + x_4 \leq 10 \\ x \geq 0 \end{cases}$$

a. Is it a convex problem? Why?

b. Do global minima exist? Why?

c. Is the global minimum unique? Why?

d. Solve the problem by means of the Frank-Wolfe method with exact line search, tolerance 10^{-6} and starting from the point $(10, 0, 0, 0)$. What is the global minimum? What is the optimal value? How many iterations are needed?

e. Solve the problem by means of the penalty method with $\tau = 0.5$, $\varepsilon_0 = 1$ and $\min(b - Ax) > -10^{-6}$ as stopping criterion. What is the global minimum? How many iterations are needed?

   Hint: at each iteration use the `fminunc` *function with the following options:*
   ```
   options = optimoptions('fminunc','GradObj','on',...
   'Algorithm','quasi-newton','Display','off').
   ```

f. Solve the problem by means of the logarithmic barrier method with $\tau = 0.5$, $\varepsilon_0 = 1$, tolerance 10^{-6} and starting from the point $(1, 1, 1, 1)$. What is the global minimum? How many iterations are needed?

   Hint: at each iteration use the `fminunc` *function with the same options as in e).*

Solutions

Exercise 1.

a. We have

$$A^+ = (A^*A)^{-1}A^* = ((Q_1R_1)^*(Q_1R_1))^{-1}(Q_1R_1)^* = (R_1^*Q_1^*Q_1R_1)^{-1}R_1^*Q_1^*$$
$$= (R_1^*R_1)^{-1}R_1^*Q_1^* = R_1^{-1}(R_1^*)^{-1}R_1^*Q_1^* = R_1^{-1}Q_1^*.$$

b.

```
function [x1, x2] = double_ls(A, b)
x1 = (A'*A) \ (A'*b);
[Q1, R1] = qr(A, 0);
pinvA = R1 \ Q1';
x2 = pinvA * b;
```

c. One has $A\hat{x} - b = 0$. Since norms are always nonnegative, then the objective function must have 0 as its minimum.

d.

```
>> xhat = [0.3; -1]
xhat =
    0.3000
   -1.0000
>> [x1, x2] = double_ls(A, b);
>> norm(x1 - xhat), norm(x2 - xhat)
ans =
   3.5704e-09
ans =
   5.9931e-13
```

The most accurate method is the second one.

e. Since $S \in \mathbb{C}^{n\times n}$, $A^+, Q^* \in \mathbb{C}^{n\times m}$, $b \in \mathbb{C}^m$,

i. Computing SQ^* costs $2mn^2$, and then computing A^+b costs $2mn$;
ii. Computing Q^*b costs $2mn$, and then computing Sc costs $2n^2$.

Exercise 2.

a. Yes, the objective function is continuous and the feasible region is closed and bounded (Weierstrass Theorem).

b. No, the objective function is not convex.

c. Yes, the Slater condition holds since the constraints are convex, $g_1(1,1) < 0$ and $g_2(1,1) < 0$.

d. No, since it does not solve the KKT system.

e. The solutions of the KKT system are:
$x = (1,2)$, $\lambda = (0,3)$;
$x = (0,2)$, $\lambda = (4,3)$;
$x = (2,2)$, $\lambda = (4,3)$.

f. $(0,2)$ and $(2,2)$ are global minima, $(1,2)$ is not a local minimum.

g. The Lagrangian dual problem is

$$\begin{cases} \max \; -\dfrac{(1+\lambda_2)^2}{\lambda_2 - 1} - \lambda_1 + 4 \\ \lambda_1 \geq 4 \\ \lambda_2 > 1 \end{cases}$$

h. No, since $\varphi(2,4) = -\infty$.

Exercise 3.

a. Yes, the hessian matrix of the objective function f is

$$Q = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$

The eigenvalues of Q are $0.7269, 2.0000, 3.1404, 6.1326$ hence f is strongly convex.

b. Yes, f is strongly convex and the feasible region is closed and convex.

c. Yes, f is strongly convex and the feasible region is closed and convex.

d. After 3 iterations the algorithm finds the approximated global minimum

$$x = 10^{-10}(0.0293, -0.0733, 0.2456, -0.0279)$$

with value $8.6828 * 10^{-11}$.

e. After 23 iterations the algorithm finds the approximated global minimum

$$x = 10^{-6}(-0.2980, -0.2980, -0.2980, -0.2980).$$

f. After 23 iterations the algorithm finds the approximated global minimum

$$x = 10^{-4}(0.4887, 0.4887, 0.4887, 0.4887).$$