Exam

Numerical Methods and Optimization course
University of Pisa, 2017-07-24

You may use Matlab, pencil or paper, or a calculator (unless explicitly stated in the exercise). You may use the quick reference sheet on Matlab's syntax posted on the web page of the course.

Exercise 1. Consider the optimization problem

$$\min_{x\in\mathbb{C}^n} \left\| \begin{bmatrix} A \\ \mu I \end{bmatrix} x - \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|^2 \tag{1}$$

with $A \in \mathbb{C}^{m\times n}, b \in \mathbb{C}^m, m > n$, and $\mu \in \mathbb{R}$, $\mu > 0$. All norms here are 2-norms (Euclidean norms), and I is the identity matrix.

a. Show that the matrix $\begin{bmatrix} A \\ \mu I \end{bmatrix}$ has full column rank (even when A does not have full column rank). *Hint*: let $z \in \mathbb{C}^n$, $z \neq 0$, and show that $\begin{bmatrix} A \\ \mu I \end{bmatrix} z$ cannot be the zero vector.

b. Show that the solution of the problem (**??**) is given by $x = (A^*A + \mu^2 I)^{-1}A^*b$.

c. Write a Matlab `function x = lsproblem(A, mu, b)` which first uses the function `chol()` to compute the Cholesky factorization $A^*A + \mu^2 I = R^*R$, with R upper triangular, and then uses it together with the previous formula to compute the solution x to problem (**??**).

   Report the code of this function on paper.

d. Use this function to compute the optimal x with

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad \mu = 0.1. \tag{2}$$

   What is the value of the objective function $\left\| \begin{bmatrix} A \\ \mu I \end{bmatrix} x - \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|^2$ for this value of x?

e. Let us consider the least-squares problem $\min\|Ax - b\|^2$ instead, with A, b again as in (**??**). Does it have a unique solution? Why?

Exercise 2. Consider the following constrained optimization problem:

$$\begin{cases} \min\ x_1^2 + x_2^2 + x_3^2 \\ x_1 \leq 1 \\ x_2 \leq 1 \\ x_3 \leq 1 \\ -x_1 - x_2 - x_3 \leq -1 \end{cases}$$

a. Do global optimal solutions exist? Why?

b. Is it a convex problem? Why?

c. Does the constraints qualification hold in any feasible point? Why?

d. Is the point $x = (1, 1, 1)$ a local minimum? Why?

e. Find all the solutions of the KKT system.

f. Find local minima and global minima.

g. Find the objective function and constraints of the Lagrangian dual problem.

h. Is $\lambda = (1, 1, 1, 0)$ an optimal solution of the Lagrangian dual problem? Why?

Exercise 3. Consider the following unconstrained optimization problem:

$$\begin{cases} \min\ \frac{5}{2}x_1^2 + \frac{5}{2}x_2^2 + x_3^2 + \frac{5}{2}x_4^2 + \frac{5}{2}x_5^2 - x_1x_4 - x_2x_5 - x_1 - 5x_2 + 5x_4 + x_5 \\ x \in \mathbb{R}^5 \end{cases}$$

a. Is it a convex problem? Why?

b. Do global minima exist? Why?

c. Is the global minimum unique? Why?

d. Solve the problem by means of the gradient method with exact line search starting from the point $(1, 1, 1, 1, 1)$ and using $\|\nabla f(x)\| < 10^{-6}$ as stopping criterion. What is the global minimum? What is the optimal value? How many iterations are needed?

e. Solve the problem by means of the gradient method with inexact line search setting $\alpha = 0.5$, $\gamma = 0.9$, $\bar{t} = 1$, starting from the point $(1, 1, 1, 1, 1)$ and using $\|\nabla f(x)\| < 10^{-6}$ as stopping criterion. What is the global minimum? How many iterations are needed?

f. Solve the problem by means of the conjugate gradient method starting from the point $(1, 1, 1, 1, 1)$ and using $\|\nabla f(x)\| < 10^{-6}$ as stopping criterion. What is the global minimum? Write the vector x found at each iteration.

Solutions

Exercise 1.

a. If $z \neq 0$, then
$$\begin{bmatrix} A \\ \mu I \end{bmatrix} z = \begin{bmatrix} Az \\ \mu z \end{bmatrix}$$
cannot be the zero vector because its second block is not zero.

b. Using the pseudoinverse:
$$x = \left(\begin{bmatrix} A \\ \mu I \end{bmatrix}^* \begin{bmatrix} A \\ \mu I \end{bmatrix} \right)^{-1} \begin{bmatrix} A \\ \mu I \end{bmatrix}^* \begin{bmatrix} b \\ 0 \end{bmatrix} = \left(\begin{bmatrix} A^* & \mu I \end{bmatrix} \begin{bmatrix} A \\ \mu I \end{bmatrix} \right)^{-1} \begin{bmatrix} A^* & \mu I \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix}$$
$$= (A^*A + \mu I \cdot \mu I)^{-1}(A^*b + \mu I \cdot 0) = (A^*A + \mu^2 I)^{-1} A^*b$$

c.

```
function x = lsproblem(A, mu, b)
[m, n] = size(A);
R = chol(A'*A + mu^2*eye(n));
x = R \ (R' \ (A' * b));
```

d.

```
>> A = ones(5,2);
>> b = [1:5]';
>> x = lsproblem(A, 0.1, b)
x =
    1.4985
    1.4985
>> norm(A*x-b)^2 + mu^2*norm(x)^2
ans =
   10.0450
```

e. A does not have full column rank, so there are infinite solutions to the problem $\min \|Ax - b\|^2$.

Exercise 2.

a. Yes, the objective function is continuous and the feasible region is closed and bounded (Weierstrass Theorem).

b. Yes, the objective function is convex and the constraints are affine.

c. Yes, the constraints are affine.

d. No, since it does not solve the KKT system.

e. The solution of the KKT system is:
$x = (1/3, 1/3, 1/3), \quad \lambda = (0, 0, 0, 2/3)$

f. $(1/3, 1/3, 1/3)$ is the global minimum.

g. The Lagrangian dual problem is

$$\begin{cases} \max \; -\frac{\lambda_1^2}{4} - \frac{\lambda_2^2}{4} - \frac{\lambda_3^2}{4} - \frac{3\lambda_4^2}{4} + \frac{\lambda_4}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 \\ \lambda \geq 0 \end{cases}$$

h. No, since $\psi(1, 1, 1, 0) = -15/4 < v(D) = 1/3$.

Exercise 3.

a. Yes, the hessian matrix of the objective function f is

$$Q = \begin{pmatrix} 5 & 0 & 0 & -1 & 0 \\ 0 & 5 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 5 & 0 \\ 0 & -1 & 0 & 0 & 5 \end{pmatrix}$$

The eigenvalues of Q are $2, 4, 4, 6, 6$ hence f is strongly convex.

b. Yes, f is strongly convex and the feasible region is closed and convex.

c. Yes, f is strongly convex and the feasible region is closed and convex.

d. After 21 iterations the algorithm finds the approximated global minimum

$$x = (6.5804e - 08, 1.0000e + 00, 2.9639e - 07, -1.0000e + 00, -6.5804e - 08)$$

with value -5.

e. After 19 iterations the algorithm finds the approximated global minimum

$$x = (5.2578e - 08, 1.0000e + 00, 1.5168e - 07, -1.0000e + 00, -5.2578e - 08).$$

f. The algorithm finds the approximated global minimum after 3 iterations.

Iteration 1:

$$x = (3.3824e - 01, 1.2206e + 00, 5.5882e - 01, -9.8529e - 01, -1.0294e - 01).$$

Iteration 2:

$$x = (-6.0261e - 02, 9.6906e - 01, 2.9316e - 01, -1.0277e + 00, 1.6287e - 03).$$

Iteration 3:

$$x = (2.7756e - 17, 1.0000e + 00, 0, -1.0000e + 00, -9.3458e - 17).$$