Exam

Numerical Methods and Optimization course
University of Pisa, 2017-09-11

You may use Matlab, pencil or paper, or a calculator (unless explicitly stated in the exercise). You may use the quick reference sheet on Matlab's syntax posted on the web page of the course.

Exercise 1. Let

$$A = \begin{bmatrix} -1 & 0 & 3 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \tag{1}$$

a. What are the eigenvalues of A?

b. Show that v_1 and v_2 are eigenvectors of A.

c. Write a Matlab `function v = power_method(A, x0, m)` that performs `m` iterations of the power method on the matrix `A`, starting from initial value `x0`, and returns the approximated eigenvector v.

d. Report on paper the output of `power_method(A, ones(4, 1), m)` for the matrix A given in (1) and the three values $m = 200, 201, 202$. Does the method converge? Why?

e. Write a Matlab `function v = subspace_iteration(A, U0, m)` that performs `m` iterations of the subspace iteration on the matrix `A`, starting from initial subspace `U0`, and returns a basis of the approximated subspace U. (Note that the dimension of the working subspace should be inferred from the size of `U0`.)

f. Report on paper the value returned by `subspace_iteration(A, [1 2; 2 3; 3 4; 4 5], 200)` on a sample run of the method. Does the method return a matrix whose columns are a basis of $\text{span}(v_1, v_2)$?

Exercise 2. Consider the following constrained optimization problem:

$$\begin{cases} \min\ -x_1^2 - x_2^2 + 4x_1 + 6x_2 \\ -x_1 \leq 0 \\ x_1^2 + x_2^2 - 6x_2 \leq 0 \end{cases}$$

a. Do global optimal solutions exist? Why?

b. Is it a convex problem? Why?

c. Does the constraints qualification hold in any feasible point? Why?

d. Find all the solutions of the KKT system.

e. Find local minima and global minima.

f. Find the objective function and constraints of the Lagrangian dual problem.

g. Find the optimal solution of the Lagrangian dual problem.

Exercise 3. Consider the following constrained optimization problem:

$$\begin{cases} \min\ 3x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1x_4 - x_2x_3 - 3x_2x_4 - x_1 - 2x_2 + 3x_3 + 4x_4 \\ 4x_1 + 3x_2 + 2x_3 + x_4 \leq 20 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 \leq 40 \\ x \geq 0 \end{cases}$$

a. Is it a convex problem? Why?

b. Do global minima exist? Why?

c. Is the global minimum unique? Why?

d. Solve the problem by means of the Frank-Wolfe method with exact line search, tolerance 10^{-6} and starting from the point $(1, 2, 1, 2)$. What is the global minimum? What is the optimal value? How many iterations are needed?

e. Solve the problem by means of the penalty method with $\tau = 0.5$, $\varepsilon_0 = 1$ and $\min(b - Ax) > -10^{-6}$ as stopping criterion. What is the global minimum? How many iterations are needed?

   Hint: at each iteration use the `fminunc` *function with the following options:*
   ```
   options = optimoptions('fminunc','GradObj','on',...
   'Algorithm','quasi-newton','Display','off').
   ```

f. Solve the problem by means of the logarithmic barrier method with $\tau = 0.5$, $\varepsilon_0 = 1$, tolerance 10^{-6} and starting from the point $(1, 2, 1, 2)$. What is the global minimum? How many iterations are needed?

   Hint: at each iteration use the `fminunc` *function with the same options as in e).*

Solutions

Exercise 1.

a. A is block triangular, so its eigenvalues are its diagonal elements $-1, 1, 2, -2$.

b. One can compute directly

$$Av_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = 2v_1, \quad Av_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ -2 \end{bmatrix} = -2v_2,$$

so v_1 and v_2 are eigenvectors of eigenvalues 2 and -2, respectively.

c.

```
function v = power_method(A, x0, k)
v = x0;
for i = 1:k
    v = A*v;
    v = v / norm(v);
end
```

d.

```
>> power_method(A, ones(4, 1), 200)
ans =
    0.5000
    0.5000
    0.5000
    0.5000
>> power_method(A, ones(4, 1), 201)
ans =
    0.5000
   -0.5000
    0.5000
   -0.5000
>> power_method(A, ones(4, 1), 202)
ans =
    0.5000
    0.5000
    0.5000
    0.5000
```

The method does not converge – it oscillates between the two values

$$\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix},$$

asymptotically. This happens because there is no single eigenvalue with maximum modulus, but there is a tie (2 and -2). So the conditions for the convergence theorem are not satisfied.

e.

```
function U = subspace_iteration(A, U0, m)
U = U0;
for i = 1:m
    U = A*U;
    [U, ~] = qr(U, 0);
end
```

f.

```
>> subspace_iteration(A, [1 2; 2 3; 3 4; 4 5], 200)
ans =
   -0.4243 0.5657
   -0.5657 -0.4243
   -0.4243 0.5657
   -0.5657 -0.4243
```

The two columns are linearly independent and they are both obtained as linear combinations of v_1 and v_2 (i.e., $-0.4243v_1 - 0.5657v_2$ and $0.5657v_1 - 0.4243v_2$); so the columns of the matrix U returned by the method form indeed a basis of $\text{span}(v_1, v_2)$.

Exercise 2.

a. Yes, the objective function is continuous and the feasible region is closed and bounded (Weierstrass Theorem).

b. No, the objective function is not convex.

c. Yes, the Slater constraints qualification holds.

d. The solutions of the KKT system are:

$x = (0, 0) \quad \lambda = (4, 1)$
$x = (0, 3) \quad \lambda = (4, 0)$
$x = (2, 3) \quad \lambda = (0, 0)$
$x = (0, 6) \quad \lambda = (4, 1)$
$x = (3, 3) \quad \lambda = (0, 1/3)$

e. The global minima are $(0, 0)$ and $(0, 6)$.

f. The Lagrangian dual problem is

$$\begin{cases} \max \ \varphi(\lambda) \\ \lambda \geq 0 \end{cases}$$

where

$$\varphi(\lambda) = \begin{cases} -\infty & \text{if } 0 \leq \lambda_2 < 1, \lambda_1 \geq 0 \\ -\infty & \text{if } \lambda_2 = 1, \lambda_1 \geq 0, \lambda_1 \neq 4 \\ 0 & \text{if } \lambda_2 = 1, \lambda_1 = 4 \\ -\dfrac{(\lambda_1 - 4)^2}{4(\lambda_2 - 1)} - 9(\lambda_2 - 1) & \text{if } \lambda_2 > 1, \lambda_1 \geq 0 \end{cases}$$

g. The optimal solution of the dual is $\lambda = (4, 1)$.

Exercise 3.

a. Yes, the hessian matrix of the objective function f is

$$Q = \begin{pmatrix} 6 & 0 & 0 & -1 \\ 0 & 4 & -1 & -3 \\ 0 & -1 & 2 & 0 \\ -1 & -3 & 0 & 4 \end{pmatrix}$$

The eigenvalues of Q are $0.5522,\ 2.2358,\ 5.7642,\ 7.4478$ hence f is strongly convex.

b. Yes, f is strongly convex and the feasible region is closed and convex.

c. Yes, f is strongly convex and the feasible region is closed and convex.

d. After 10 iterations the algorithm finds the approximated global minimum

$$x = (0.1667, 0.5000, 0, 0)$$

with value -0.5833.

e. After 21 iterations the algorithm finds the approximated global minimum

$$x = (0.1667, 0.5000, 0.0000, 0.0000).$$

f. After 23 iterations the algorithm finds the approximated global minimum

$$x = (0.1677, 0.5008, 0.0000, 0.0001).$$